Note:- If $\{a\}$ and $\{b\}$ are singletons, we will write

$F^{-1}(a)$ for $F(\{a\})$, $F^{-1}(b)$ for $F(\{b\})$ اذا المجموعة تتكون من عنصر واحد سوف نكتب الاقواس بها

Ex.: ① Let $f: A \to B$, $A = \{a, b, c, d\}$

$B = \{1, 2, 3, 4\}$

$f(a) = 1$, $f(b) = 3$, $f(c) = 2$, $f(d) = 4$.

الصورة المباشرة لـ A $F(A) = \{\overset{f(a)}{1}, \overset{f(b)}{3}, \overset{f(c)}{2}, \overset{f(d)}{4}\}$

معكوس الصورة $F^{-1}(B) = \{a, b, c, d\}$

② let $f: Z \to \mathbb{R}$, $f(x) = x^2 + 3$

$A = \{1, 0, 5, -6, 2\} \subset Z$

$B = \{3, 4, 25, 19, 9, 2\} \subset R$.

$F(A) = \{f(1), f(0), f(5), f(-6), f(2)\}$

$= \{\overset{1+3}{4}, \overset{0+3}{3}, 28, 39, 7\}$

$F^{-1}(B) = \{x \in Z : x^2 + 3 \in B\}$ الصورة العكسية يعني اي عنصر صورة 3 بعمل هذه الدالة

$= \{x^2+3=3,\ x^2+3=4,\ x^2+3=25,\ x^2+3=19,\ x^2+3=9,\ x^2+3=2\}$

$F^{-1}(B) = \{0, 1, -1, -4, 4\}$

$x^2+3=3 \Rightarrow x^2=0 \Rightarrow x=0 \in Z$.

$x^2+3=25 \Rightarrow x^2=22 \Rightarrow x=\mp\sqrt{22} \notin Z$

$x^2+3=4 \Rightarrow x^2=1 \Rightarrow x=\mp 1 \in Z$

$x^2+3=19 \Rightarrow x^2=16 \Rightarrow x=\mp 4 \in Z$

$x^2+3=9 \Rightarrow x^2=6$

$\Rightarrow x=\mp\sqrt{6} \notin Z$

$x^2+3=2 \Rightarrow x^2=-1$

$\Rightarrow x=\mp\sqrt{-1} \notin Z$

3- Let $f : \mathbb{R}-\{0\} \longrightarrow \mathbb{R}$ be a function

$f(x) = \frac{1}{x}$, $\forall\, x \in \mathbb{R}-\{0\}$, $C = (0,1]$

مجموعة جزئية

$F(C) = \{ y \in \mathbb{R} : 1 \leq y < \infty \}$

Theorem:- Let $f : X \to Y$ be a function and $A \subseteq X$, $B \subseteq X$, if $A = B$ then $F(A) = f(B)$.

Proof:- suppose that $A = B$ and let $y \in F(A)$

$\Rightarrow \exists\, x \in A$ s.t. $y = f(x)$ $\qquad F(C) = \{ y \in B , \exists x \in C \text{ s.t. } y = f(x) \}$

$\because A = B \Rightarrow x \in B$ s.t. $y = f(x)$

$\Rightarrow f(x) \in F(B)$ s.t. $y = f(x)$

$\Rightarrow y \in F(B) \Rightarrow F(A) \subseteq F(B)$ --- ①

let $z \in F(B)$

$\Rightarrow \exists\, e \in B$ s.t $z = f(e)$

$\because A = B \Rightarrow \exists\, e \in A$ s.t. $z = f(e)$

$\Rightarrow f(e) \in F(A)$ s.t. $z = f(e) \Rightarrow z \in F(A)$.

$\therefore F(B) \subseteq F(A)$ --- ②

From ① and ② we get $F(A) = F(B)$.

Ex.:- to show that the converse of this theorem is not true.

Sol.:- No, since

let $f: \mathbb{Z} \to \mathbb{R}$, $f(x) = x^2$

$A = \{2, 1, 3\}$, $B = \{2, 1, -3\}$

$F(A) = \{4, 1, 9\}$. $F(B) = \{4, 1, 9\}$

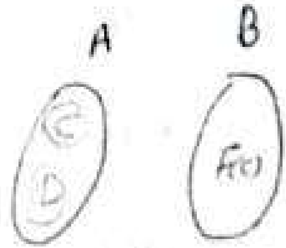
$F(A) = F(B)$ but $A \neq B$.

Theorem:- Let $f: A \to B$ be a function and if $C \subseteq A$, $D \subseteq A$, then

① $F(C \cup D) = F(C) \cup F(D)$.

② $F(C \cap D) \subset F(C) \cap F(D)$.

③ $F(C) - F(D) \subset F(C - D)$.



Proof(1):- let $y \in F(C \cup D)$, $y \in B$, since f is a function

$\Rightarrow \forall y \in F(C \cup D), \exists x \in (C \cup D)$ s.t $y = f(x)$

$\Rightarrow \forall y \in F(C \cup D), \exists x \in C$ or $x \in D$ s.t $y = f(x)$

$\Rightarrow f(x) \in F(C)$ or $f(x) \in F(D)$

$\Rightarrow y \in F(C)$ or $y \in F(D)$

$\Rightarrow y \in F(C) \cup F(D)$

$\therefore F(C \cup D) \subseteq F(C) \cup F(D)$ --- ①

$\Leftarrow$ H.w

Proof(2):- let $y \in F(C \cap D)$, $y \in B$

$\because f$ is a function

$\Rightarrow \forall y \in F(C \cap D)$, $\exists x \in (C \cap D)$ s.t $y = f(x)$

$\Rightarrow \forall y \in F(C \cap D)$, $\exists x \in C$ and $x \in D$, $y = f(x)$

صورة x تنتمي F(c)

$\Rightarrow f(x) \in F(C)$ and $f(x) \in F(D)$

$\Rightarrow y \in F(C)$ and $y \in F(D)$

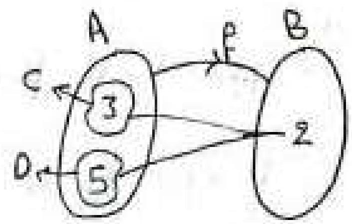
$\Rightarrow y \in F(C) \cap F(D)$.

$\therefore F(C \cap D) \subset F(C) \cap F(D)$.

* To show that $F(C) \cap F(D) \not\subset F(C \cap D)$.

let $f: A \to B$, $A = \{3, 5\}$, $B = \{2\}$

and $f(x) = 2$, $\forall x \in A$



suppose that $C = \{3\}$, $D = \{5\}$

$C \cap D = \emptyset$

$F(C \cap D) = F(\emptyset) = \emptyset$

$F(C) = \{2\}$, $F(D) = \{2\}$

$\Rightarrow F(C) \cap F(D) = \{2\}$

$\Rightarrow \{2\} \not\subset \emptyset$

$\therefore F(C) \cap F(D) \not\subset F(C \cap D)$.

Proof (3):- let $y \in F(C) - F(D)$, $y \in B$

$\Rightarrow y \in F(C)$ and $y \notin F(D)$

$\because$ f is a function

$\Rightarrow \exists\, x \in A$ s.t. $y = f(x)$

$\Rightarrow \exists\, x \in C$ and $x \notin D$ s.t. $y = f(x)$

$\Rightarrow \exists\, x \in (C - D)$, s.t $y = f(x)$

$\Rightarrow f(x) \in F(C - D)$, s.t. $y = f(x)$

$\Rightarrow y \in F(C - D)$.

$\therefore F(C) - F(D) \subset F(C - D)$.

* To show that $F(C-D) \not\subset F(C) - F(D)$.

نفس المثال السابق

Sol:- let $f: A \to B$, $A = \{3, 5\}$, $B = \{2\}$,

and $f(x) = 2$, $\forall\, x \in A$

suppose that $C = \{3\}$, $D = \{5\}$

$C - D = \{3\}$

$F(C-D) = F(3) = \{2\}$

$F(C) - F(D) = \{2\} - \{2\} = \emptyset$

$\therefore F(C-D) = \{2\} \not\subset \emptyset = F(C) - F(D)$.

$\therefore F(C-D) \not\subset F(C) - F(D)$.

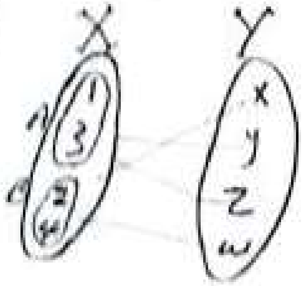
Ex: $F(A\cap B) \not\subset F(A)\cap F(B)$

$F(A-B) \not\subset F(A)-F(B)$.

Sol: $X=\{1,2,3,4\}$, $Y=\{x,y,z,w\}$

$A=\{1,3\}$, $B=\{2,4\}$

$F(A)=\{z,y\}$, $F(B)=\{x,z\}$



$F(A)\cap F(B)=\{z\}$

$A\cap B=\emptyset$, $F(A\cap B)=\emptyset$

$\therefore F(A)\cap F(B) \not\subset F(A\cap B)$.

$A-B=\{1,3\} \Rightarrow F(A-B)=\{z,y\}$

$F(A)-F(B)=\{y\}$ $\therefore F(A-B)\not\subset F(A)-F(B)$.

Ex: $f: R\to R$, $f(x)=x^2+1$, $\forall x\in R$ and

$D=\{17\}$ or $D=\{5,10\}$, find $f^{-1}(D)$.

Sol: $F^{-1}(D)=F^{-1}(\{17\})=\{x\in R: f(x)\in D\}$

$=\{x\in R: x^2+1=17\}$

$=\{x\in R: x^2=16\}=\{-4,4\}$.

$F^{-1}(D)=F^{-1}(\{5,10\})=\{x\in R: f(x)\in D\}$

$=\{x\in R: x^2+1=5 , x^2+1=10\}$

$=\{x\in R: x^2=4 , x^2=9\}$

$=\{-2,2,-3,3\}$.

Theorem:- Let $f: X \to Y$ be a function and $C \subset Y$, $D \subset Y$, if $C = D$, then $f^{-1}(C) = f^{-1}(D)$.

Proof:- let $x \in f^{-1}(C)$, $x \in X$

$\because$ f is a function

$\Rightarrow f(x) \in C$

$\because C = D \Rightarrow f(x) \in D$

$\Rightarrow x \in f^{-1}(D)$ $\quad \therefore f^{-1}(C) \subset f^{-1}(D)$ --- ①

Now, let $y \in f^{-1}(D)$, $y \in Y$

$\because$ f is a function

$\Rightarrow f(y) \in D$

$\because C = D \Rightarrow f(y) \in C$

$\Rightarrow y \in f^{-1}(C)$ $\quad \therefore f^{-1}(D) \subset f^{-1}(C)$ --- ②

from ① and ② we get

$f^{-1}(C) = f^{-1}(D)$.

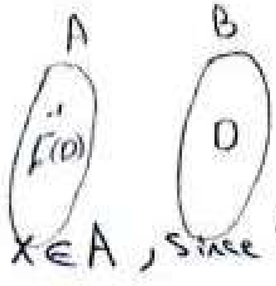
Theorem:- Let $f: A \to B$ be a function and $C \subseteq B$, $D \subseteq B$, then

$F^{-1}(D) = \{x \in A : f(x) \in D\}$

① $F^{-1}(C \cup D) = F^{-1}(C) \cup F^{-1}(D)$. H.W

② $F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$.

③ $F^{-1}(C - D) \overset{=}{\cancel{\subseteq}} F^{-1}(C) - F^{-1}(D)$.



Proof (2):- let $x \in F^{-1}(C \cap D)$, $x \in A$, since f is fun.

$\Rightarrow f(x) \in C \cap D$

$\Rightarrow f(x) \in C$ and $f(x) \in D$

$\Rightarrow x \in F^{-1}(C)$ and $x \in F^{-1}(D)$

$\therefore F^{-1}(C \cap D) \subset F^{-1}(C) \cap F^{-1}(D)$ --- ①

by the same way $F^{-1}(C) \cap F^{-1}(D) \subset F^{-1}(C \cap D)$ --- ②

From ① and ② we get

$F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$.

Proof(3):- let $x \in f^{-1}(C-D)$, $x \in A$, f is function

$\Rightarrow f(x) \in (C-D)$

$\Rightarrow f(x) \in C$ and $f(x) \notin D$

$\Rightarrow x \in f^{-1}(C)$ and $x \notin f^{-1}(D)$

$\therefore x \in [f^{-1}(C) - f^{-1}(D)]$

$\Rightarrow f^{-1}(C-D) \subset f^{-1}(C) - f^{-1}(D)$.

4) let $B = \{1,2,3,4\}$ and $C = \{a,b,c\}$. List fore different surjective functions from B to C.

$f_1 = \{(1,a),(2,a),(3,b),(4,c)\}$

$f_2 = \{(1,b),(2,b),(3,a),(4,c)\}$

$f_3 = \{(1,c),(2,a),(3,b),(4,a)\}$

$f_4 = \{(1,a),(2,b)\ (3,c),(4,b)\}$