15^th^ Lecture

Example (5.2): If $f:(X,\tau)\to(X^*,\tau^*)$ and (X,τ) is a discrete topological space, (X^*,τ^*) any topological space. Then f is continuous on X.

Solution:

f continuous at $a\in X$ iff $\forall^{open} G^*\ni f(a), \exists^{open} G\ni a, f(G)\subset G^*$

Example (5.1): Let $X=\{a,b,c,d,e\}$, $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},X\}$, and $X^*=\{x,y,z,u\}$, $\tau^*=\{\emptyset,\{x\},\{y\},\{x,y\},\{x,y,z\},X\}$.

Let $f:X\to X^*$ be defined as follows:

$f(a)=x, f(b)=f(c)=y, f(d)=z, f(e)=z$

Determine whether f is continuous at the point a,b,c,d,e

Solution: $a\in X$, $f(a)=x$

The open sets G^* containing $f(a)=x$ are $\{x\}$, $\{x,y\}$, $\{x,y,z\}$, X^*

The open sets G containing a are $\{a\}$, $\{a,b\}$, X

We have $f(\{a\})=\{f(a)\}=\{x\}\subset G^*, \forall\, G^*\ni f(a)$

$\Rightarrow\ \exists^{open}\, G=\{a\}\ni a, \forall^{open}\, G\ni f(a), f(G)\subset G^*, \forall\, G^*\ni f(a)$

$\Rightarrow\ f$ is continuous at $a\in X$

$b\in X, f(b)=y$

The open sets G^* containing $f(b)=y$ are $\{y\}$, $\{x,y\}$, $\{x,y,z\}$, X^*

The open sets G containing b are $\{b\}$, $\{a,b\}$, X

We have $f(\{b\})=\{f(b)\}=\{y\}\subset G^*, \forall\, G^*\ni f(b)$

$\Rightarrow\ \exists^{open}\, G=\{b\}\ni b, \forall^{open}\, G\ni f(b), f(G)\subset G^*, \forall\, G^*\ni f(b)$

$\Rightarrow\ f$ is continuous at $b\in X$

$c\in X, f(c)=y$

The open sets G^* containing $f(c)=y$ are $\{y\}$, $\{x,y\}$, $\{x,y,z\}$, X^*

The open set G containing c is X

We have $f(\{c\})=\{f(c)\}=\{y\}\subset G^*, \forall\, G^*\ni f(c)$

$\Rightarrow\ \exists^{open}\ G = \{c\} \ni c, f(G) \not\subset G^*$

$f(x) = \{x, y, z\} \not\subset G^*$

$[\forall\ G^* \ni y, \exists\ X \ni c\ , f(x) \not\subset G^*]$

$d \in X, f(d) = z$

The open sets G^* containing $f(d) = z$ are $\{x, y, z\}$, X^*

The open sets G containing d is X

We have $f(\{d\}) = \{f(d)\} = \{z\} \subset G^*, \forall\ G^* \ni f(d)$

$\Rightarrow\ \exists^{open}\ G = \{d\} \ni d, \forall^{open}\ G \ni f(d), f(G) \subset G^*, \forall\ G^* \ni f(d)$

$\Rightarrow\ f$ is continuous at $d \in X$

$e \in X, f(e) = z$

The open sets G^* containing $f(e) = z$ are $\{x, y, z\}$, X^*

The open set G containing e is X

We have $f(\{e\}) = \{f(e)\} = \{z\} \subset G^*, \forall\ G^* \ni f(e)$

$\Rightarrow\ \exists^{open}\ G = \{e\} \ni e, \forall^{open}\ G \ni f(e), f(G) \subset G^*, \forall\ G^* \ni f(e)$

$\Rightarrow\ f$ is continuous at $e \in X$

Remark (5.1): If f is continuous at each point $x \in X$. Then we say that f is continuous on X.

Theorem (5.1): (The Fundamental Theorem of Continuity)

If $f: (X, \tau) \to (X^*, \tau^*)$. Then the continuity of f on X is equivalent to each of the following conditions:

(1) The inerse image of each open set in X^* is open in X.

(2) The inverse image of each closed set in X^* is closed in X.

(3) $f(\bar{A}) \subset \overline{f(A)}, \forall\ A \subset X$.

Proof: Continuity of $f \leftrightarrow f^{-1}(G^*)$ is open in X, $\forall^{open}\ G^* \subset X^*$

Continuity of $f \leftrightarrow f^{-1}(G^*)$ is closed in X, $\forall^{closed}\ G^* \subset X^*$

Suppose that f is continuous on X

We need to show that $f^{-1}(G^*)$ is open in X, $\forall^{open}\, G^* \subset X^*$

Let $G^* \subset X^*$ be any open we have to prove $f^{-1}(G^*)$ is open

Let $x \in f^{-1}(G^*)$ be any point

$\Rightarrow\ f(x) \in G^*$, but $G^* \subset X^*$ is given open and $f(x) \in G^*$

Since f is continuous

$\Rightarrow\ \forall^{open}\, G^* \ni f(x), \exists^{open}\, G \ni x, f(G) \subset G^*$

$\Rightarrow\ f(x) \in f(G) \subset G^* \Rightarrow\ x \in G \subset f^{-1}(G^*)$

$\Rightarrow\ \forall\ x \in f^{-1}(G^*), \exists^{open}\, G \ni x, G \subset f^{-1}(G^*)$

$\Rightarrow\ f^{-1}(G^*)$ is open in X

Conversly: Suppose that $f^{-1}(G^*)$ is open in X, $\forall^{open}\, G^* \subset X^*$

We need to show that f is continuous on X

Let $G^* \subset X^*$ be any open set $\Rightarrow\ f^{-1}(G^*) \subset X$ is open

But by using the definition of open set we set

$\forall\, x \in f^{-1}(G^*), \exists^{open}\, G \ni x, x \in G \subset f^{-1}(G^*)$

$\Rightarrow\ \forall\, x \in f^{-1}(G^*), \exists^{open}\, G \ni x, f(x) \in f(G) \subset G^*\ \Rightarrow\ f(x) \in f(G) \subset G^*$

$\Rightarrow\ \forall^{open}\, G^* \subset X^*, \exists^{open}\, G, f(G) \subset G^* \Rightarrow\ f$ is continuous on X

Now (1) $\leftrightarrow$ (2)

Suppose the inverse image of each open set in X^* is open in X

We need to show that the inverse image of each closed set in X^* is closed in X

Let $F^* \subset X^*$ be any closed set

$\Rightarrow\ {F^*}^c = X^* - F^*$ is open set

$\Rightarrow\ f^{-1}({F^*}^c)$ is open $\Rightarrow\ \left(f^{-1}(F^*)\right)^c \Rightarrow\ f^{-1}(F^*)$ is closed in X

Conversely: Suppose that the inverse of image of each closed set in X^* is closed in X.

We need to show that the inverse image of each open set in X^* is open in X

$\Rightarrow\ f^{-1}({G^*}^c)$ is closed in $X\ \Rightarrow\ \left(f^{-1}(G^*)\right)^c$ is closed in X

$\Rightarrow\ f^{-1}(G^*)$ is open in X

Now we prove (2) $\leftrightarrow$ (3)

Suppose that the inverse image of each closed set in X^* is closed in X

We need to show that $f(\bar{A}) \subset \overline{f(A)}, \forall\, A \subset X$

Since $f(A) = f(A), \forall\, A \subset X$ and $f(A) \subset \overline{f(A)}$ closed

$\Rightarrow\; A \subset f^{-1}(\overline{f(A)})$ closed

$\Rightarrow\; \bar{A} \subset \overline{f^{-1}\big(\overline{f(A)}\big)} = f^{-1}\big(\overline{f(A)}\big)$

$\Rightarrow\; \bar{A} \subset f^{-1}\big(\overline{f(A)}\big) \;\Rightarrow\; f(\bar{A}) \subset \overline{f(A)}$

Conversely: suppose that $f(\bar{A}) \subset \overline{f(A)}, \forall\, A \subset X$

We need to show that $f^{-1}(F^*)$ is closed in $X, \forall^{closed}\, F \subset X^*$

Let $F^* \subset X^*$ be any closed set

We need to prove that $f^{-1}(F^*)$ is closed in X

Let $E = f^{-1}(F^*)$

We are given $f(\bar{E}) \subset \overline{f(E)} = \overline{f\big(f^{-1}(F^*)\big)} = \overline{F^*} = F^* = f(E)$

$\Rightarrow\; f(\bar{E}) \subset f(E)$ but $E \subset \bar{E} \;\Rightarrow\; f(E) \subset f(\bar{E})$

$\Rightarrow\; f(\bar{E}) = f(E)$

$\Rightarrow\; f^{-1}(f(\bar{E})) = f^{-1}\big(f(E)\big)$

$\Rightarrow\; \bar{E} = E = f^{-1}(F^*) \;\Rightarrow\; f^{-1}(F^*)$ is closed in X.
