1st Lecture

Chapter One

الفضاءات التبولوجية

Topological Spaces

Definition (1.1): Let $X \neq \emptyset$ and τ be a family of subsets of X then we say that τ is topology on X (or (X, τ) is a topological space) if:

(1) $\emptyset, X \in \tau$.

(2) If $A_i \in \tau$, $\forall\, i \in \mathbb{N} \;\Rightarrow\; \bigcup\limits_{i\in\mathbb{N}} A_i \in \tau$.

(The union of any number of element of τ belong to τ)

(3) If $A_i \in \tau$, $\forall\, i = 1,2, \dots, n$ then $\bigcap_{i=1}^{n} A_i \in \tau$.

(The finite intersection of elements of τ belong to τ)

The sets $A_i \in \tau$ called the open sets of the topological space (X, τ).

Example (1.1): Let $X \neq \emptyset$, $\tau = \{\emptyset, X\}$. Then τ is a topology on X and (X, τ) is a topological space.

Proof:

(i) $\emptyset \in \tau$ and $X \in \tau \;\Rightarrow\; \emptyset, X \in \tau$

(ii) Since $\emptyset \cup X = X \cup \emptyset = X \in \tau \;\Rightarrow\; X \cup \emptyset \in \tau$

(iii) Since $\emptyset \cap X = X \cap \emptyset = \emptyset \in \tau \;\Rightarrow\; X \cup \emptyset \in \tau$

Therefore τ is a topology on X.

ضعيفة

Remark (1.1): The topology τ in the above example is called the **weak** (indiscrete) topology on X.

مجموعة كل المجموعات الجزئية من X

Example (1.2): Let $X \neq \emptyset$ and $\tau = \{A : A \subset X\}$ then (X, τ) is a topological space.

Proof:

(i) Since $\emptyset \subset X \;\Rightarrow\; \emptyset \in \tau \;\Rightarrow\; \emptyset, X \in \tau$

Since $X \subset X \Rightarrow X \in \tau$

(ii) Let $A_i \in \tau$, $i \in \mathbb{N} \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \tau$.

$\Rightarrow A_i \subset X$, $i \in \mathbb{N}$

$\Rightarrow \bigcup_{i\in\mathbb{N}} A_i \subset X \Rightarrow \bigcup_{i\in\mathbb{N}} A_i \in \tau$

(iii) If $A_i \in \tau$, $\forall i = 1,2, \dots , n$

$\Rightarrow A_i \subset X$, $\forall i = 1,2, \dots , n$

$\Rightarrow \bigcap_{i=1}^{n} A_i \subset X \Rightarrow \bigcap_{i=1}^{n} A_i \in \tau$

Therefore τ is a topology on X.

مبعثرة

Remark (1.2): The topology τ in the above example is called the **discrete** topology on X.

Example (1.3): Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$

Determine whether τ is a topology on X or no.

Solution:

(i) We have $\emptyset, X \in \tau$.

(ii) We have $\{a\} \cup \{b, c\} = \{a, b, c\} \in \tau$

$\{a\} \cup \{a, b, c\} = \{a, b, c\} \in \tau$

$\{b, c\} \cup \{a, b, c\} = \{a, b, c\} \in \tau$

Therefore the union of any number of element of τ is in τ.

(iii) We have $\{a\} \cap \{b, c\} = \emptyset \in \tau$

$\{a\} \cap \{a, b, c\} = \{a\} \in \tau$

$\{b, c\} \cap \{a, b, c\} = \{b, c\} \in \tau$

Thus the intersection of any number of elements of τ is in τ.

Hence τ is a topology on X.

Example (1.4): Let (X, d) be a metric space then the family of all open subsets of X with respect to (w.r.t.) d forms a topology on X

Proof:

Let $\tau = \{G : G \subset X \; is \; open \;\; w.r.t. \;\; d\}$

(i) Since (X, d) is a metric space

$\Rightarrow \;\; \emptyset, X$ are open in X w.r.t. d

$\Rightarrow \;\; \emptyset, X \in \tau$

(ii) If $G_i \subset X$ is open w.r.t. $d \; \forall \; i \in \lambda$ but (X, d) is a metric space

$\Rightarrow \;\; \bigcup_{i \in \lambda} G_i$ is open w.r.t. d

$\Rightarrow \;\; \bigcup_{i \in \lambda} G_i \in \tau$

(iii) If $G_i \subset X$ is open w.r.t. d, $\forall \; 1 \leq i \leq n$ but (X, d) is a metric space

$\Rightarrow \;\; \bigcap_{i=1}^{n} G_i$ is open in X

$\Rightarrow \;\; \bigcap_{i=1}^{n} A_i \in \tau$

Hence τ is a topology on X.

الاعتيادي

Remark (1.3): The topology τ in the above example is called the **usual** topology on X.

Example (1.5): Let $X = \mathbb{N}$ be the set of natural numbers. Let τ be a family of all subsets of $\mathbb{N}$ of the form $\{1,2, \ldots, n\}$ with $\emptyset$ and X. Prove that τ is a topology on X.

Proof:

(i) We are given $\emptyset, X \in \tau$.

(ii) Let $A_1 = \{1,2, \ldots, n_1\}$ and $A_2 = \{1,2, \ldots, n_2\}$ be elements of τ.

We have

$$A_1 \cup A_2 = \{1,2, \ldots, n_1\} \cup \{1,2, \ldots, n_2\} = \begin{cases} \{1,2, \ldots, n_1\}, & if \quad n_1 \geq n_2 \\ \{1,2, \ldots, n_2\}, & if \quad n_2 \geq n_1 \end{cases}$$

In both cases we have

$A_1 \cup A_2 \in \tau$

In general let $A_i \in \tau, \forall\, i \in \lambda$

We have $\bigcup_{i\in\lambda} A_i = A_j$ where $j = \max_{i\in\lambda} \{i\}$

But $A_j \in \tau \;\Rightarrow\; \bigcup_{i\in\lambda} A_i \in \tau$

(iii) We have

$$A_1 \cap A_2 = \{1,2,\dots,n_1\} \cap \{1,2,\dots,n_2\} = \begin{cases} \{1,2,\dots,n_1\}, & if \quad n_1 \le n_2 \\ \{1,2,\dots,n_2\}, & if \quad n_2 \le n_1 \end{cases}$$

In general let $A_i \in \tau, \forall\, i \in \lambda$

We have $\bigcap_{i=1}^{n} A_i = A_j$ where $j = \min_{i\in\lambda} \{i\}$

But $A_j \in \tau \;\Rightarrow\; \bigcap_{i=1}^{n} A_i \in \tau$

Thus (X, τ) is a topological space.

Theorem (1.1): Intersection of a family of topological spaces on a set is a topological space on this set.

Proof:

Let $\tau = \bigcap_{\alpha\in\Lambda} \tau_\alpha$, τ_α be a topology $\forall\, \alpha \in \Lambda$

(1) Since τ_α is a topology on X, $\forall\, \alpha \in \Lambda$ then $\emptyset$ and X are belong to τ_α, $\forall\, \alpha \in \Lambda$. Thus $\emptyset$ and X are belong to $\bigcap_{\alpha\in\Lambda} \tau_\alpha$. Thus $\emptyset, X \in \tau$.

(2) Let $A_\beta \in \tau, \forall\, \beta \in \xi \;\Rightarrow A_\beta \in \bigcap_{\alpha\in\Lambda} \tau_\alpha$

Thus $A_\beta \in \tau_\alpha, \forall\, \alpha \in \Lambda, \forall\, \beta \in \xi$

Since τ_α is a topology $\forall\, \alpha \in \Lambda$, then $\bigcup_{\beta\in\xi} A_\beta \in \tau_\alpha, \forall\, \alpha \in \Lambda$.

Thus $\bigcup_{\beta\in\xi} A_\beta \in \bigcap_{\alpha\in\Lambda} \tau_\alpha \;\Rightarrow\; \bigcup_{\beta\in\xi} A_\beta \in \tau$.

(3) Let $B_j \in \tau, \forall\, j = 1,2,\dots,n \;\Rightarrow B_j \in \bigcap_{\alpha\in\Lambda} \tau_\alpha, j = 1,2,\dots,n$

$\Rightarrow$ Thus $B_j \in \tau_\alpha, \forall\, \alpha \in \Lambda, \forall\, j = 1,2,\dots,n$

$\Rightarrow \bigcap_{j=1}^{n} B_j \in \tau_\alpha, \forall\, \alpha \in \Lambda$

$\Rightarrow \quad \bigcap_{j=1}^{n} B_j \in \bigcap_{\alpha \in \Lambda} \tau_\alpha \quad \Rightarrow \quad \bigcap_{j=1}^{n} B_j \in \tau$

(1) , (2) and (3) implies that τ is a topology on X.

Exercises (1.1): (Homework)

(1) Let $X \neq \emptyset$ and τ contains the empty set and all subsets of X whose complement is finite w.r.t. X. Then τ is a topology on X.

(2) Let $X = \{a, b, c\}$ and $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_3 = \{\emptyset, \{a\}, \{b\}, X\}$. Determine which one is represents a topology on X.
