The Inverse Trigonometric Functions

1- $y = \sin^{-1} x$

The inverse sin function, is defined to be the inverse of the restricted Sine function

$$\sin x \quad , \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

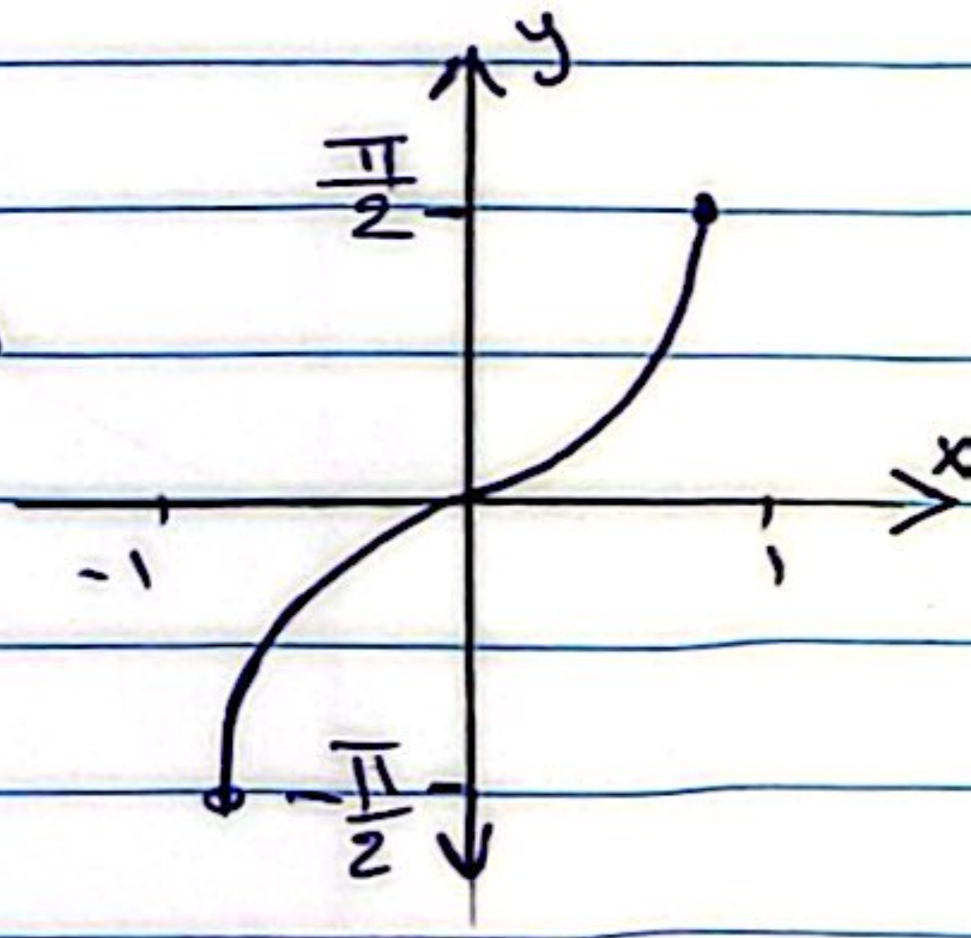
$D_y = [-1, 1]$ or $D_y = \{x : x \in \mathbb{R}, -1 \leq x \leq 1\}$

$R_y = [-\frac{\pi}{2}, \frac{\pi}{2}]$ or $R_y = \{y : y \in \mathbb{R} ; -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$

$\because \quad \sin^{-1}(\sin x) = x \quad ; \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\sin(\sin^{-1} x) = x \quad ; \text{if } -1 \leq x \leq 1$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \quad ; -1 < u < 1$$

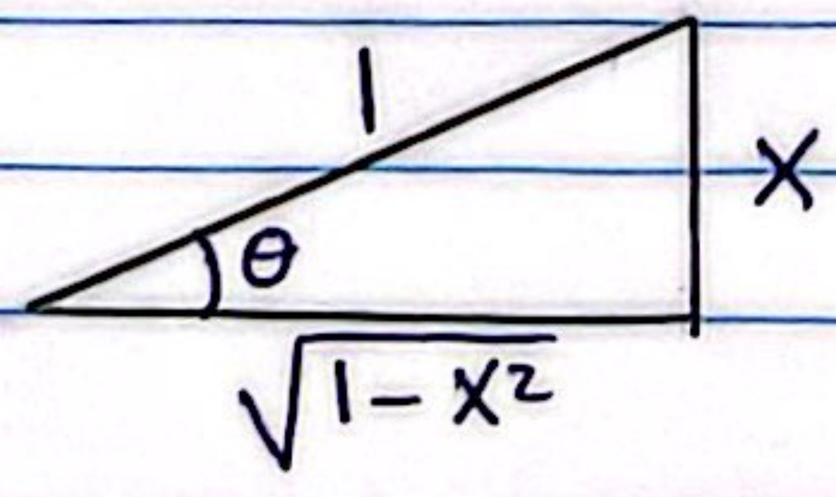


$$\int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + c$$

Note:- $\frac{1}{\sin x} = \sin^{-1} x$ دالة مثلثية

But

$(\sin x)^{-1} \neq \sin^{-1} x$ دالة مثلثية عكسية



2- $y = \cos^{-1} x$.

The inverse cosine function, is defined to be the inverse of the restricted cosine function

$\cos x \quad ; \quad 0 \leq x \leq \pi$

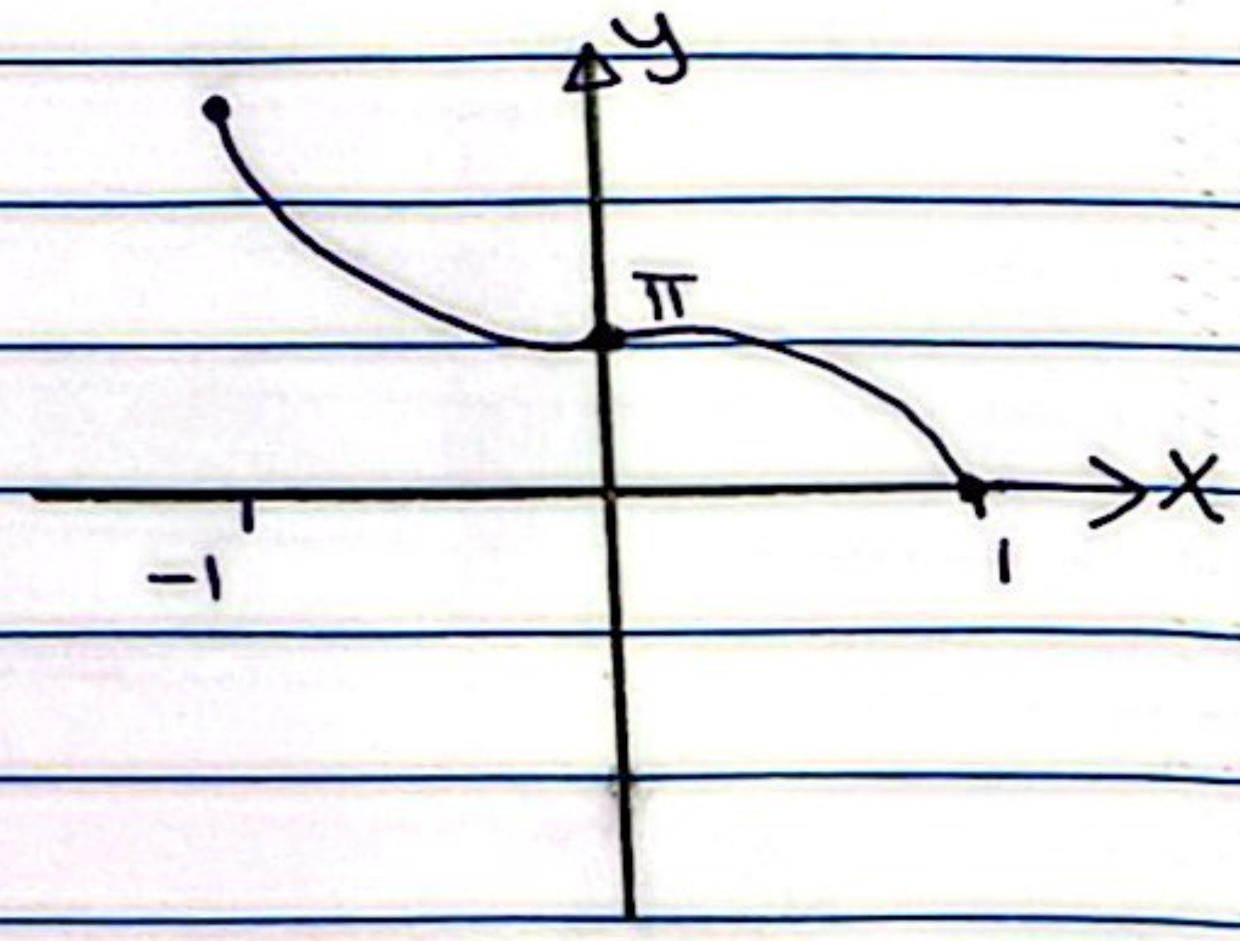
$D_y = [-1, 1] \quad ; \quad R_y = [0, \pi]$

$\cos^{-1}(\cos x) = x \quad ; \text{ if } 0 \leq x \leq \pi$

$\cos(\cos^{-1} x) = x \quad \text{if} \quad -1 \leq x \leq 1$

$$\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx} \quad ; -1 < u < 1$$

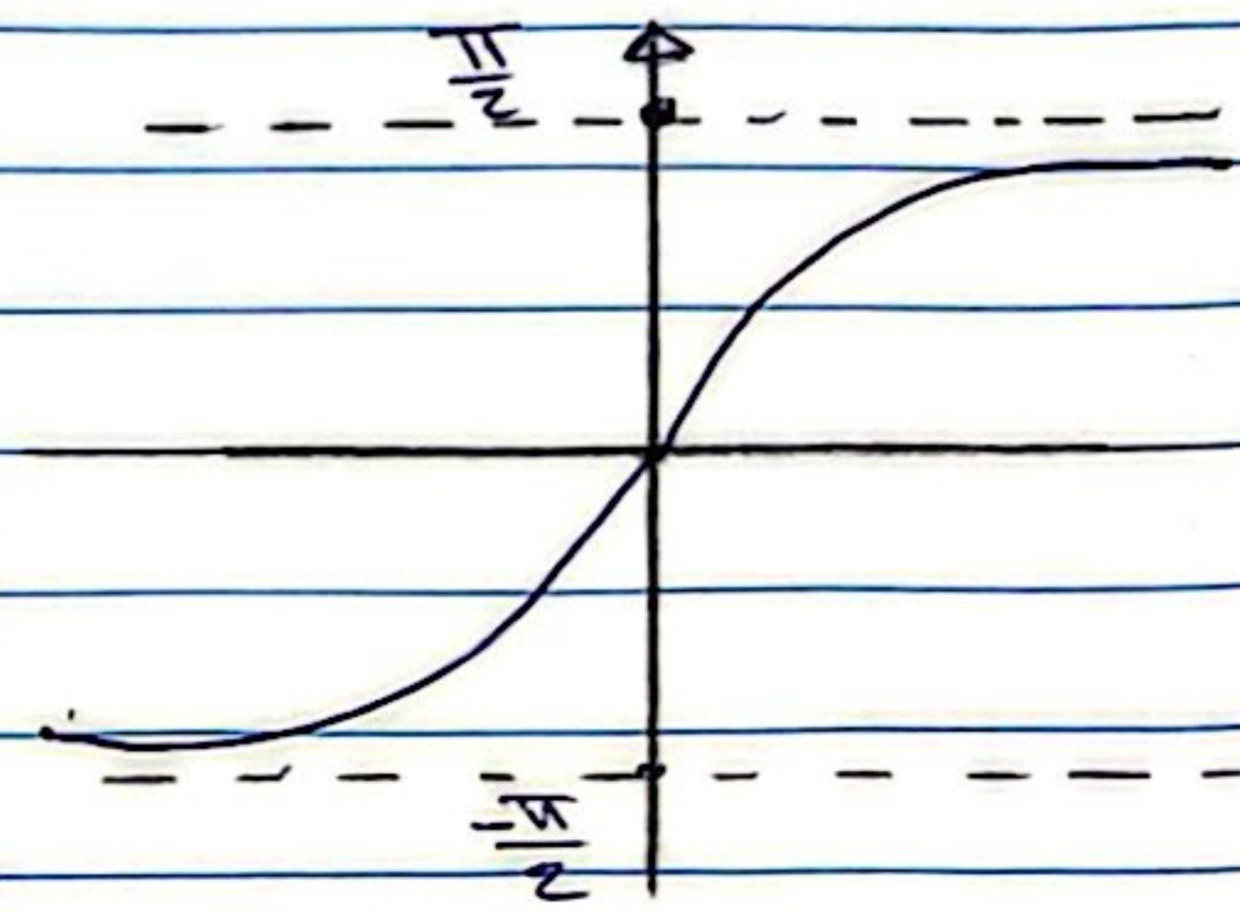
$$\int \frac{du}{\sqrt{1-u^2}} = -\cos^{-1} u + c$$



3- $y = \tan^{-1} x$

The inverse tangent function, is defined to be the inverse of the restricted tangent function

$\tan x \quad ; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

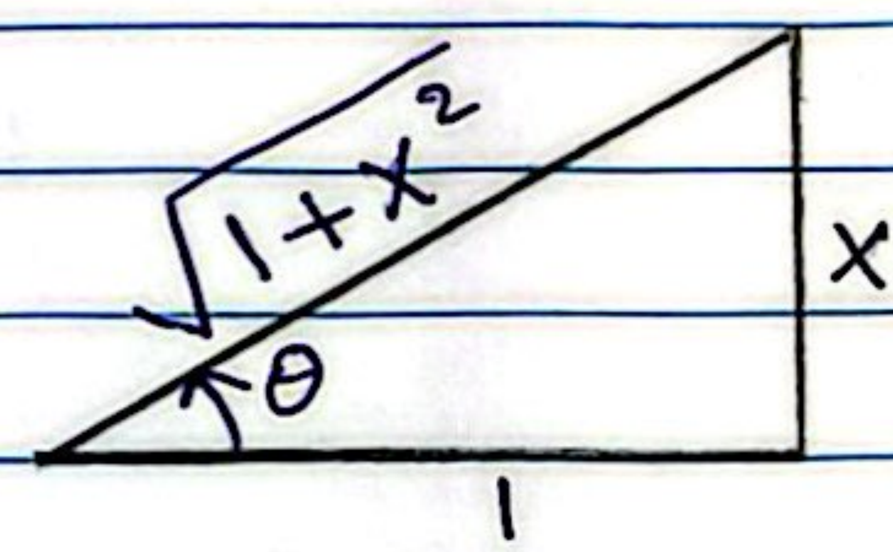


$D_y = (-\infty, \infty) \quad ; \quad R_y = (-\frac{\pi}{2}, \frac{\pi}{2})$

$\tan(\tan^{-1} x) = x \quad \text{if} \quad -\infty < x < \infty$

$\tan^{-1}(\tan x) = x \quad \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

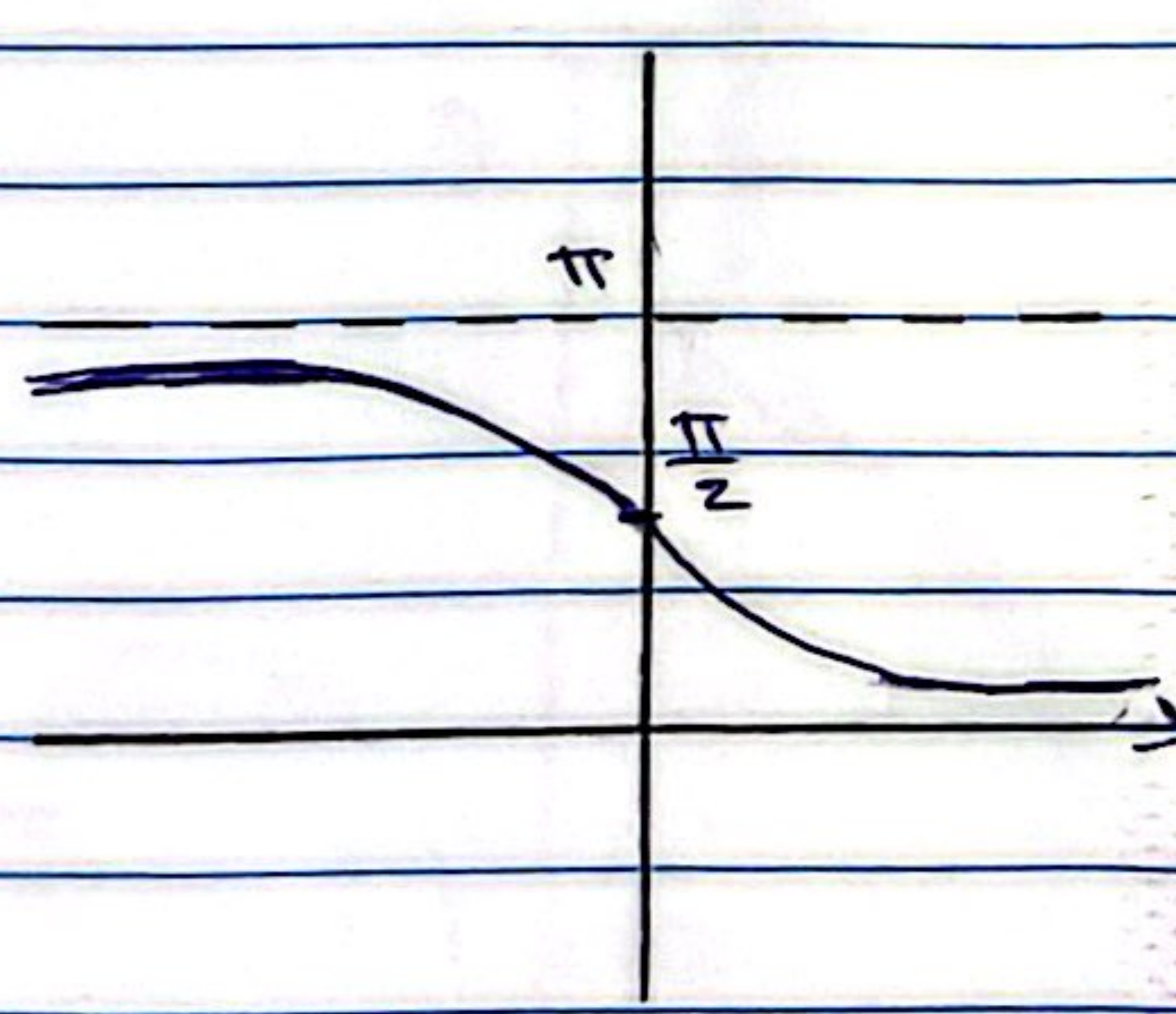
$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx} \quad ; \quad (-\infty < u < \infty)$$



$$\int \frac{1}{1+u^2} du = \tan^{-1} u + c$$

4- $y = \cot^{-1} x$

$\cot x \quad ; \quad 0 < x < \pi$



$D_y = (-\infty, \infty) \quad ; \quad R_y = (0, \pi)$

$\cot^{-1}(\cot x) = x \quad \text{if} \quad 0 < x < \pi$

$\cot(\cot^{-1} x) = x \quad \text{if} \quad -\infty < x < \infty$

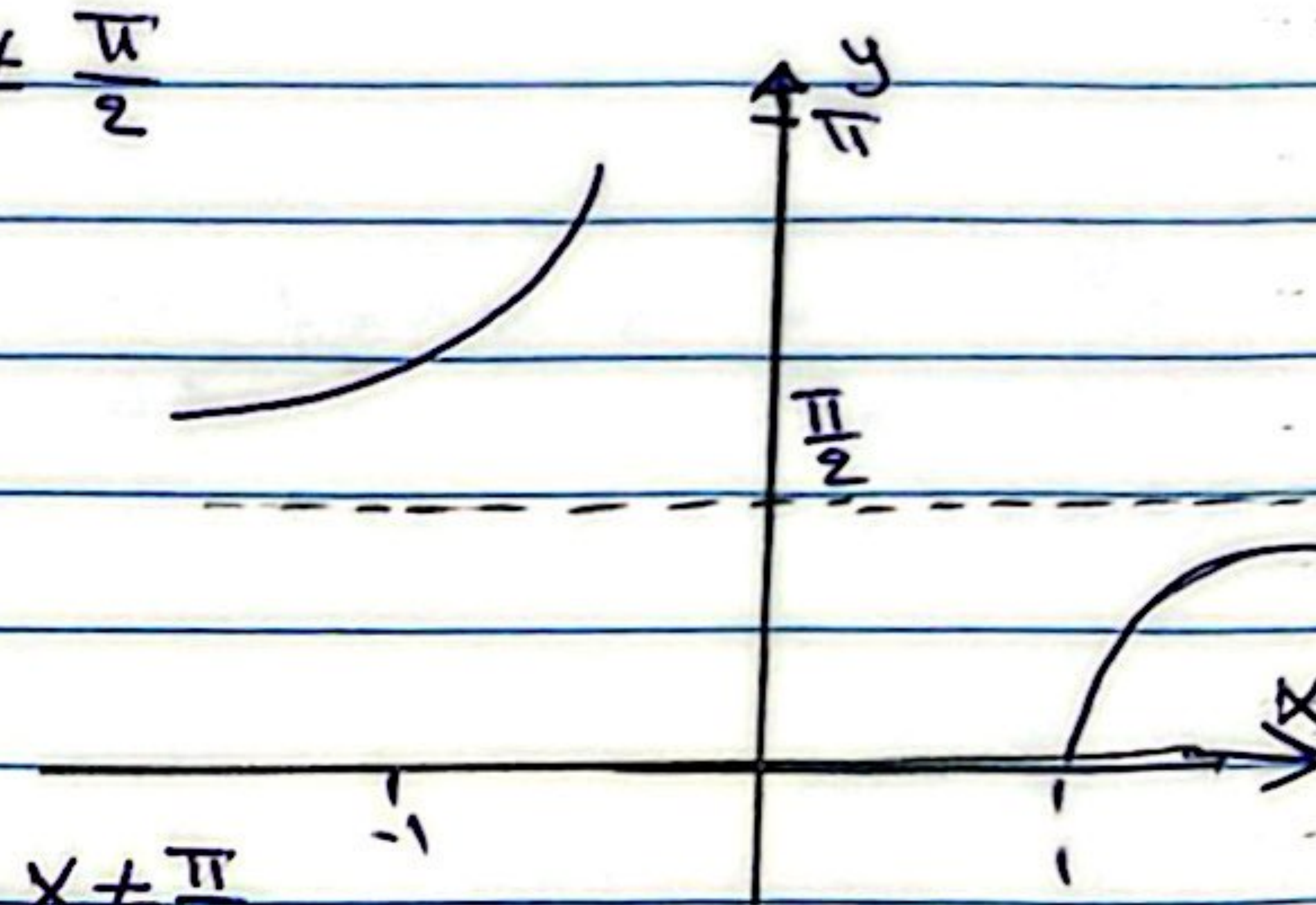
$$\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx} \quad ; \quad -\infty < u < \infty$$

$$\int \frac{du}{1+u^2} = -\cot^{-1} u + c$$

5- $y = \sec^{-1} x$

The inverse secant function is defined to be the inverse of the restricted secant function

$\sec x$, $0 \le x \le \pi$ with $x \neq \frac{\pi}{2}$



$D_y = (-\infty, -1] \cup [1, \infty)$

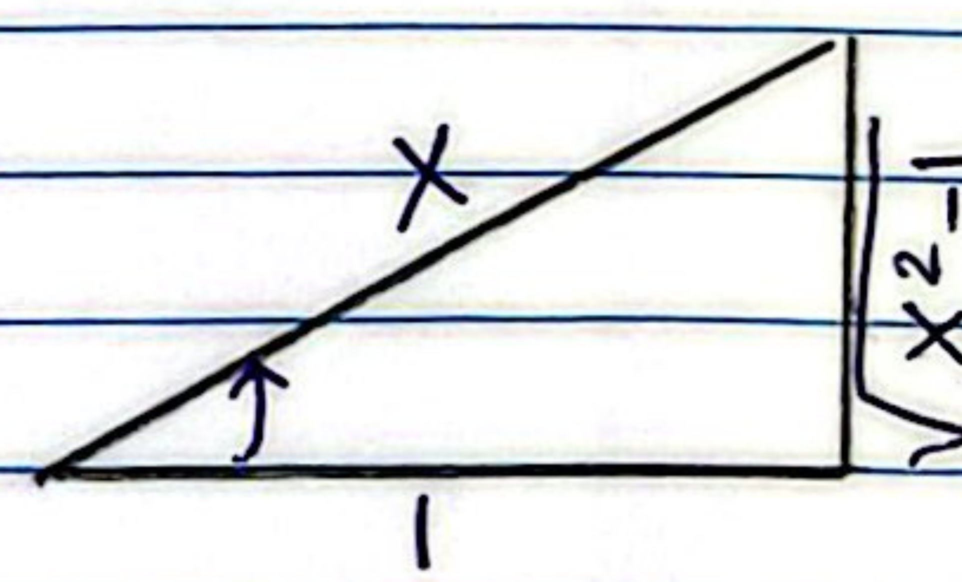
$R_y = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

$\sec^{-1}(\sec x) = x$, if $0 \le x \le \pi$, $x \neq \frac{\pi}{2}$

$\sec(\sec^{-1} x) = x$, if $|x| \geq 1$

$$\frac{d}{dx} \sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} , |u| > 1$$

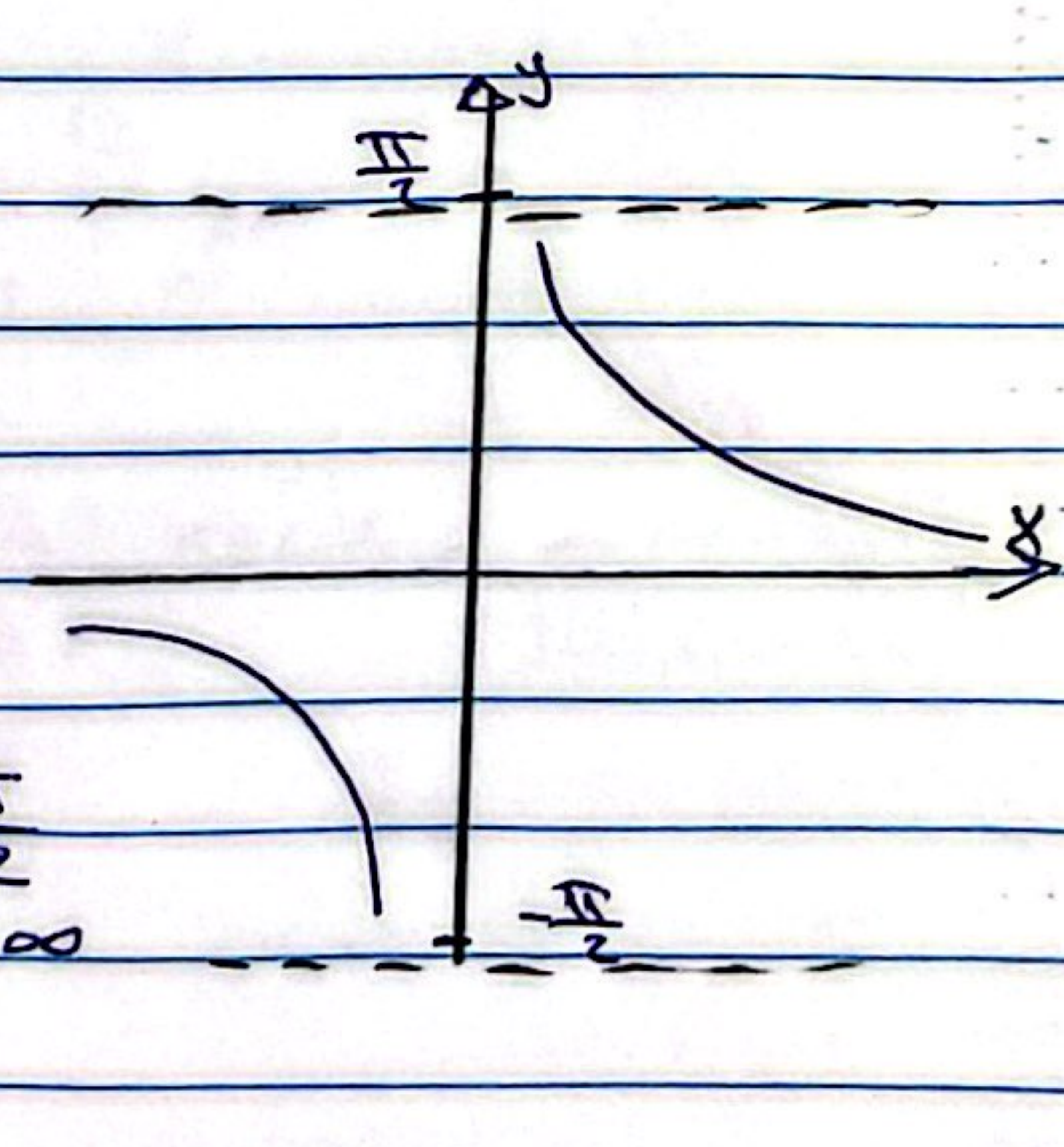
$$\int \frac{1}{u\sqrt{u^2-1}} du = \sec^{-1}|u| + c$$



6- $y = \csc^{-1} x$

$\csc x$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$

$D_y = (-\infty, \infty)$; $R_y = (-\frac{\pi}{2}, \frac{\pi}{2})$



$\csc^{-1}(\csc x) = x$, if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\csc(\csc^{-1} x) = x$ if $-\infty < x < \infty$

$$\frac{d}{dx} \csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx} , -\infty < u < \infty$$

$$\int \frac{1}{u\sqrt{u^2-1}} = -\csc^{-1} u + c$$

Ex Find y' for the following

1. $y = \sin^{-1}(x^2)$; $\frac{\partial}{\partial x}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$$y' = \frac{2x}{\sqrt{1-x^4}}$$

2. $y = \tan^{-1}(4x)$; $\frac{\partial}{\partial x}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

$$y' = \frac{4}{1+(4x)^2} = \frac{4}{1+16x^2}$$

3. $y = 7\cos^{-1}\sqrt{2x}$; $\frac{\partial}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$

$$y' = 7 \cdot \frac{-1}{\sqrt{1-(\sqrt{2x})^2}} \frac{1}{2}(2x)^{-\frac{1}{2}} \cdot 2$$

$$= \frac{-7}{\sqrt{2x}\,\sqrt{1-2x}}$$

4. $y = \tan^{-1} e^x$

$$y' = \frac{1}{1+(e^x)^2} \cdot e^x = \frac{e^x}{1+e^{2x}}$$

5. $y = \sec^{-1}(\ln x)$; $\frac{\partial}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

$$y' = \frac{1}{|\ln x|\sqrt{(\ln x)^2-1}} \cdot \frac{1}{x}$$

6. $y = \sin^{-1}(\cos x)$

$$y' = \frac{1}{\sqrt{1-(\cos x)^2}} \cdot -\sin x = \frac{-\sin x}{\sin x} = -1$$

7. $y = \cos^{-1}(e^x + 1)$

$$y' = \frac{-1}{\sqrt{1-(e^x+1)^2}} \cdot e^x = \frac{-e^x}{\sqrt{\cancel{1} - e^{2x} - 2e^x \cancel{+1}}} = \frac{-e^x}{\sqrt{-e^{2x} - 2e^x}}$$

8. $xy + \tan^{-1}(xy) = 3$

$$(x y' + y \cdot 1) + \frac{1}{1+(xy)^2}(xy' + y \cdot 1) = 0$$

$$xy' + y + \frac{xy' + y}{1+(xy)^2} = 0$$

$$xy' + y = -\frac{xy' + y}{1+(xy)^2}$$

$$\therefore -xy' + y = xy' + xy'(xy)^2 + y + y(xy)^2$$

$$-xy' - xy' - xy'(xy)^2 = \cancel{-y} + y(xy)^2 \cancel{-y}$$

$$y'(+2x + x(xy)^2) = -y(xy)^2 - 2y$$

$$y' = \frac{-y(xy)^2 - 2y}{+2x + x(xy)^2}$$

9. $\sin^{-1} xy - e^{xy} = 5$

$$\frac{xy' + y\cdot 1}{\sqrt{1-(xy)^2}} - e^{xy}(xy' + y\cdot 1) = 0$$

$$\frac{xy'}{\sqrt{1-(xy)^2}} + \frac{y}{\sqrt{1-(xy)^2}} = xy'e^{xy} + ye^{xy}$$

$$\frac{xy'}{\sqrt{1-(xy)^2}} - xy'e^{xy} = ye^{xy} - \frac{y}{\sqrt{1-(xy)^2}}$$

$$y'\left[\frac{x}{\sqrt{1-(xy)^2}} - xe^{xy}\right] = ye^{xy} - \frac{y}{\sqrt{1-(xy)^2}}$$

$$\therefore y' = \frac{ye^{xy} - \frac{y}{\sqrt{1-(xy)^2}}}{\left(\frac{x}{\sqrt{1-(xy)^2}} - xe^{xy}\right)}$$

$$= \frac{+y\cancel{\left(e^{xy} - \frac{1}{\sqrt{1-(xy)^2}}\right)}}{-x\cancel{\left(e^{xy} - \frac{1}{\sqrt{1-(xy)^2}}\right)}} = \frac{-y}{x}$$

H.W.

Find y' for the following functions:-

1. $\sec^{-1}(e^{xy}) + \frac{x}{y} = 2$

2. $\tan^{-1}y + \tan^{-1}x = 0$

3. $\sin^{-1}(\ln xy) = \sqrt{xy}$

4. $y = \cos^{-1}(2x+1)$

5. $y = (\tan x^2)^{-1}$

6. $y = (\sec^{-1}(2x))^2$

7. $y = e^x \csc^{-1}x$

8. $y = \cos^{-1}(e^x)$

9. $y = \sin^{-1}(\frac{1}{x}) + \cos^{-1}(\frac{1}{x})$

Ex

Evalute the following integrals:-

1. $\int \frac{1}{\sqrt{9-x^2}} dx = \int \frac{1}{\sqrt{9(1-\frac{x^2}{9})}} dx$

$= \int \frac{1}{3\sqrt{1-(\frac{x}{3})^2}} dx = \sin^{-1}(\frac{x}{3}) + c$

2. $\int \frac{\sin x}{1+\cos^2 x} dx = -\tan^{-1}(\cos x) + c$

3. $\int \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1}(e^x) + c$

4. $\int \frac{1}{x\sqrt{9x^2-1}} = \sec^{-1}(3x) + c$

5. $\int \frac{1}{\sqrt{2-x^2}}\,dx = \int \frac{1}{\sqrt{2(1-(\frac{x}{\sqrt{2}})^2}}\,dx = \sin^{-1}\frac{x}{\sqrt{2}} + c$

6. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = \sin^{-1}(\tan x) + c$

7. $\int \frac{dx}{\sin^2 x \csc^2 x} = \int \frac{\csc^2 x}{(1+\cot^2 x)}\,dx$

$= -\tan^{-1}(\cot x) + c$

or $= -(-\cot^{-1}(\cot x) + c = x + c$

8. $\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx = \sin^{-1}(\ln x) + c$

9. $\int \frac{1}{x\sqrt{9x^2-1}}\,dx = \sec^{-1}(3x) + c$

10. $\int \frac{1}{\sec x + \sin x \tan x}\,dx = \int \frac{dx}{\frac{1}{\cos x} + \sin x + \frac{\sin x}{\cos x}}$

$$= \int \frac{\cos x}{1+\sin^2 x}\, dx$$

$$= \tan^{-1}(\sin x) + C$$

Ex

Evalute the definite integrals :-

1. $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\, dx = \tan^{-1} x \Big|_{-1}^{1}$

$$= \tan^{-1}(+1) - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

2. $\displaystyle\int_{1}^{3} \frac{1}{\sqrt{x}\,(1+x)}\, dx = 2\tan^{-1}\sqrt{x} \Big|_{1}^{3}$

$$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1}(1)\right]$$

$$= 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{6}$$

Hint :-

$[\tan^{-1}(1) = y \quad]$ اخذ tan الطرفين

$\tan(\tan^{-1}(1)) = \tan y$

$\therefore \; 1 = \tan y$

$\therefore \; y = \frac{\pi}{4}$