scalar multiplication

(S1) $\cdot : R \times R^n \longrightarrow R^n$

Let $x \in R^n$, $x = (x_1, x_2, \ldots, x_n)$

$\alpha \cdot x = (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$

$\because x_i \in R$ and $\alpha \in R$, $\alpha x_i \in R$ ~~first~~

forall $1 \leq i \leq n$

so $\alpha \cdot x \in R^n$

R^n is closed under scalar multiplication

(S2)

$$\alpha \cdot (x+y) = \alpha \left[(x_1, x_2 \ldots x_n) + (y_1, y_2, \ldots, y_n) \right]$$

$$= \alpha \left[(x_1+y_1, x_2+y_2, \ldots x_n+y_n) \right]$$

$$= (\alpha(x_1+y_1), \alpha(x_2+y_2), \ldots, \alpha(x_n+y_n))$$

$$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \ldots, \alpha x_n + \alpha y_n)$$

$$= (\alpha x_1, \alpha x_2, \ldots \alpha x_n) + (\alpha y_1, \alpha y_2, \ldots \alpha y_n)$$

$$= \alpha (x_1, x_2, \ldots, x_n) + \alpha (y_1, y_2, \ldots, y_n)$$

$$= \alpha x + \alpha y$$

forall $x, y \in R^n$, $\alpha \in R$

S_3 $(\alpha+\beta).X = (\alpha+\beta)(x_1, x_2, \dots x_n)$

$= ((\alpha+\beta)x_1, (\alpha+\beta)x_2, \dots (\alpha+\beta)x_n)$

$= (\alpha x_1+\beta x_1, \alpha x_2+\beta x_2, \dots \alpha x_n+\beta x_n)$

$= (\alpha x_1, \alpha x_2, \dots \alpha x_n) + (\beta x_1, \beta x_2, \dots \beta x_n)$

$= \alpha.(x_1, x_2, \dots x_n) + \beta.(x_1, x_2, \dots x_n)$

$= \alpha.X + \beta.X$

for all $\alpha, \beta \in R$, $X \in R^n$

S_4 $(\alpha \cdot \beta) \cdot X = (\alpha\beta)(x_1, \dots x_n)$

$= ((\alpha\beta) \cdot x_1, (\alpha\beta)x_2, \dots (\alpha\beta)x_n)$

$= (\alpha(\beta . x_1), \alpha(\beta x_2), \dots \alpha(\beta x_n))$

$= \alpha.(\beta x_1, \beta x_2, \dots \beta x_n)$

$= \alpha(\beta \cdot (x_1, \dots x_n))$

$= \alpha(\beta \cdot X)$

for all $\alpha, \beta \in R$, $X \in R^n$

S_5 Since $1 \in R$ we have

$1 \cdot X = 1 \cdot (x_1, x_2, \dots x_m)$

$= (1 \cdot x_1, 1x_2, \dots, 1x_m)$

$= (x_1, x_2 \dots, x_n)$

$= X$

similarly $X \cdot 1 = X$ for all $x \in R$

So $(R^n, +, \cdot, R)$ is a vector space over R.

Some properties of vector space

Lemma

Let X be a vector space over a field K. Then

1. $0 \cdot x = \theta$
2. $\alpha \cdot \theta = \theta$
3. $-(\alpha \cdot x) = (-\alpha) \cdot x = \alpha(-x)$
4. for any $x, y \in X$, then there exists one element $z \in X$ such that $x + z = y$
5. $\alpha(x - y) = \alpha x - \alpha y$

where $\alpha, 0 \in K$, $x, y \in X$

proof:

1. $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$

$$0 \cdot x = 0 \cdot x + 0 \cdot x$$
$$0 \cdot x + {}^{-}0 \cdot x = 0 \cdot x$$
$$\theta = 0 \cdot x$$

2. $\alpha \cdot \theta = \alpha \cdot (\theta + \theta)$

$$= \alpha \cdot \theta + \alpha \cdot \theta$$
$$\alpha \cdot \theta + {}^{-}\alpha \cdot \theta = \alpha \cdot \theta$$
$$\theta = \alpha \cdot \theta$$

(3) $-(\alpha \cdot x) = (-\alpha)x = \alpha(-x)$

$(\alpha + -\alpha)x = 0 \cdot x = 0$

$\alpha \cdot x + (-\alpha)x = 0$

$(-\alpha)x = -\alpha \cdot x$

$\alpha \cdot (x + -x) = \alpha \cdot 0 = 0$

$\alpha \cdot x + \alpha(-x) = 0$

$\alpha(-x) = -\alpha \cdot x$ $\quad \forall x, y \in X$

$\alpha \in K$

$\therefore \Rightarrow -\alpha \cdot x = (-\alpha) \cdot x = \alpha(-x)$

(4) Let $x, y \in X$, if $x = y$ then $z = 0$ is the only element $x + 0 = y$

if $x \neq y$ then let $z = y - x$

So $x + z = x + y - x = y$

let $z_1 \in X$ such that

$x + z_1 = y \longrightarrow z_1 = y - x = z$

So z is the uniqe element make $x + z = y$

(5) $\alpha(x - y) = \alpha(x + (-1)y)$ $\quad \forall x, y, \in X$

$= \alpha \cdot x + \alpha((-1)y)$

$= \alpha \cdot x + (\alpha \cdot -1)y$

$= \alpha \cdot x + -\alpha y$

$= \alpha x - \alpha y$

$\forall x, y \in X$

$\alpha \in K$

(14)

التوليف الخطي الترتيب قاعدة

Linear Combination ~~(...)~~

A linear combination of vectors $x_1, x_2, \dots x_n$ of a vector space X is an expression of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

or we can write it

$$\sum_{i=1}^{n} \alpha_i x_i$$

where ~~...~~ the coefficents $\alpha_1, \alpha_2, \dots, \alpha_n \in K$ are any scalars

Example let C^2 be vector space over R

Then $x_1 = (1, 2+i)$

$x_2 = (1+i, i)$ where $i = \sqrt{-1}$

let $\alpha_1 = 2$, $\alpha_2 = 1$

find $\alpha_1 x_1 + \alpha_2 x_2$

Solution

$$\begin{aligned}\alpha_1 x_1 + \alpha_2 x_2 &= 2(1, 2+i) + 1(1+i, i) \\ &= (2\cdot 1, 2\cdot(2+i)) + (1\cdot(1+i), 1\cdot i) \\ &= (2, 4+2i) + (1+i, i) \\ &= (2+(1+i), (4+2i)+i) \\ &= (3+i, 4+3i)\end{aligned}$$

span set المجموعة المولدة

For any nonempty subset $M \subseteq X$, the set of all linear combinations of vectors of M is called the span of M written ($\operatorname{span} M$)

$$\operatorname{Span} M = \{ x \in X : x = \sum_{i=1}^{n} \alpha_i e_i , n \in \mathbb{N}, \alpha_i \in K, e_i \in M \}$$

$M = \{ e_1, e_2, \ldots, e_n, \ldots \}$

we say that M is span X if

$$X = \operatorname{span} M$$

that is mean every element in X can be written as linear combination of elements of M.

خطياً مستقلة معتمدة

Linearly independence (dependence)

A set of vector $M=\{x_1, \dots x_n\}, n\geq 1$ in a vector space X over K is called linearly independent if

خطياً مستقلة

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n = 0$$

implies

$$\alpha_1 = \alpha_2 = \dots \alpha_n = 0$$

~~and~~ If some of $\alpha_i \neq 0$, at least one of then then the set M is called linearly dependent معتمدة خطياً

Note

Let $M \subseteq X$, $M=\{x_1, \dots x_n\}, n\geq 1$ is linearly dependent. then at least one of the vectors x_i, ~~...~~ can be written as a linear combination of the others.

$$\alpha_1 x_1 + \dots \alpha_i x_i + \alpha_n x_n = 0$$

$\because \alpha_i \neq 0$, so

$$-\alpha_i x_i = \alpha_1 x_1 + \dots \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_i + \dots \alpha_n x_n$$

$$x_i = \frac{-\alpha_1}{\alpha_i} x_1 + \dots \frac{\alpha_{i-1}}{-\alpha_i} x_{i-1} + \frac{\alpha_{i+1}}{-\alpha_i} x_i + \frac{\alpha_n}{-\alpha_i} x_n$$

قاعدة

basis (Hamel basis)

قاعدة هامل

A sub set $M \subseteq X$ where X is a vector space over K is called basis if M is $\underline{\text{span } X}$ and $\underline{\text{linearly independent}}$

(1) شرط

(2) شرط

Example Let R^3 is vector space over R is $M = \{(1,0,0), (0,2,0), (0,0,1)\}$ basis for R^3.

Solution to show that span $M = R^3$

Let $x \in R^3$, so $x = (x_1, x_2, x_3)$ when $x_1, x_2, x_3 \in R$, now

$\cancel{x = (x_1, x_2, x_3)}$

$$\alpha_1(1,0,0) + \alpha_2(0,2,0) + \alpha_3(0,0,1) =$$

$$(\alpha_1,0,0) + (0,2\alpha_2,0) + (0,0,\alpha_3) = (\alpha_1, 2\alpha_2, \alpha_3)$$

if we take $\alpha_1 = x_1$

$2\alpha_2 = x_2 \rightarrow \alpha_2 = \frac{x_2}{2} = \frac{1}{2} \cdot x_2$

$\alpha_3 = x_3$

$$x_1(1,0,0) + \tfrac{1}{2}x_2(0,2,0) + x_3(0,0,1) = (x_1, x_2, x_3) = X$$

$\therefore$ span $M = R^3$

to show linear independent.

Let $\alpha_1, \alpha_2, \alpha_3 \in K$

$\alpha_1(1,0,0) + \alpha_2(0,2,0) + \alpha_3(0,0,1) = (0,0,0)$

$(\alpha_1,0,0) + (0,2\alpha_2,0) + (0,0,\alpha_3) = (0,0,0)$

$(\alpha_1, 2\alpha_2, \alpha_3) = (0,0,0)$

$\alpha_1 = 0$

$2\alpha_2 = 0 \longrightarrow \alpha_2 = 0$

$\alpha_3 = 0$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$

so M is linearly independ

$\therefore$ M is a basis for R^3

Note finite and infinite dimension vector space

A vector space $\overline{X}$ is said to be finite dimensional if it have finite basis, the number of element the basis is called dimension of X and denoted by dim(X). if it basis is not finite set then we called $\overline{X}$ to be infinite dimensional.

Example

Let $X = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a, b, c, d \in R \}$

$(X, +, \cdot, R)$ is a vector space

Find the basis for it

Solution

Let $M = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \}$

Let $A \in X$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in R$

we can write A

$$\cancel{A} \; a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}b\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \frac{d}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

So ~~space~~ any element in X can be written as linearly combination of element M.

$$\alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 & 2\alpha_2 \\ \alpha_3 & 2\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\alpha_1 = 0$

$2\alpha_2 = 0 \rightarrow \alpha_2 = 0$

$\alpha_3 = 0$

$2\alpha_4 = 0 \rightarrow \alpha_4 = 0$

So $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

M is linearly independent

So M is a basis for X

$\dim(X) = 4$

Theorem

Let X be a finite dimension vector space over k, Then every basis for X has the same number of element.

that is mean if M_1, M_2, M_3 is a basises of X and X is finite dimension then

~~dim(M_1) = dim(M_2) = dim~~

number of element M_1 = number of element M_2

= number of element M_3

Theorem

Every vector space has a basis.