proposition Let X be an pre Hilbert space over the field K then

① $\langle \alpha x + \beta y , z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

② $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$

③ $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$

for all $x, y, z \in X$, $\alpha, \beta \in K$

proof

① $\langle \alpha x + \beta y, z \rangle = \langle \alpha x, z \rangle + \langle \beta y, z \rangle$ use IP_1

$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ use IP_2

② $\langle x, \alpha y \rangle = \overline{\overline{\langle x, \alpha y \rangle}}$ since $\bar{\bar{x}} = x$

$= \overline{\langle \alpha y, x \rangle}$

$= \overline{\alpha \langle y, x \rangle}$

$= \overline{\langle y, x \rangle}\, \bar{\alpha}$

$= \bar{\alpha}\, \overline{\langle y, x \rangle}$ since K is field it is $\overline{\langle y, x \rangle}, \bar{\alpha} \in K$ and it is commutative

$= \bar{\alpha} \langle x, y \rangle$

③ $\langle x, \alpha y + \beta z \rangle = \overline{\overline{\langle x, \alpha y + \beta z \rangle}}$

$= \overline{\langle \alpha y + \beta z, x \rangle}$ by using ①

$= \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle}$

$= \bar{\alpha}\, \overline{\langle y, x \rangle} + \bar{\beta}\, \overline{\langle z, x \rangle}$

$= \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$

Corollarly Let X be an inner product space over $\mathbb{K}$ then the following properties are satisfies

(1) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(2) $\langle x-y, z\rangle = \langle x, z\rangle - \langle y, z\rangle$

(3) $\langle x, y-z\rangle = \langle x, y\rangle - \langle x, z\rangle$

proof: (1) by using proposition (3) and $\alpha=1$, $\beta=1$ we get

$\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

(3) similarly using proposition (3) and $\alpha=1$, $\beta=-1$ we get

$\langle x, y-z\rangle = \langle x, y\rangle - \langle x, z\rangle$

to proof (2) by using proposition (1) $\alpha=1$, $\beta=-1$

$\langle x-y, z\rangle = \langle x, z\rangle - \langle y, z\rangle$

proposition

Let X be an inner product space over field K

Then

(1) $\langle 0, y\rangle = 0$, $\langle x, 0\rangle = 0$

(2) ~~If $\langle x, z\rangle = \langle y, z\rangle$ then $x=y$ and for all $x, y, z \in X$~~

~~Proof~~

then

$$\langle 0, y\rangle = \langle 0+0, y\rangle$$
$$= \langle 0, y\rangle + \langle 0, y\rangle$$

by add $-\langle 0, y\rangle$ to both side

$$\cancel{\langle 0, y\rangle} - \cancel{\langle 0, y\rangle} = \langle 0, y\rangle + \cancel{\langle 0, y\rangle} - \cancel{\langle 0, y\rangle}$$

0 $\qquad$ 0

$$= \langle 0, y\rangle + 0$$

$\therefore \langle 0, y\rangle = 0$

② ~~let $\langle x, z\rangle =$~~ similarly we prove that

$\langle x, 0\rangle = 0$

~~② let $\langle x, z\rangle = \langle y, z\rangle$~~

~~$\langle x, z\rangle - \langle y, z\rangle = 0$~~

~~$\langle x - y, z\rangle = 0$~~

~~now~~ **Proposition** every inner product on vector space X over field k is a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle} \geq 0$.

$$\|x\|^2 = \langle x, x\rangle$$

proof: let $x \in X$, then

$\|x\| = \sqrt{\langle x, x\rangle}$ since $\langle x, x\rangle \geq 0$

so $\|x\| = \sqrt{\langle x, x\rangle} \geq 0$

$\therefore \|x\| \geq 0$

for all $x \in X$

(2) $\|x\| = 0 \longleftrightarrow \sqrt{\langle x, x\rangle} = 0$

$$\langle x, x\rangle = 0$$

From the properties of inner product we get

$$x = 0$$

(3) Let $\alpha \in K$

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle}$$

$$= \sqrt{\alpha \langle x, \alpha x\rangle}$$

$$= \sqrt{\alpha \bar{\alpha} \langle x, x\rangle} \qquad \because \alpha\bar{\alpha} = |\alpha|^2$$

$$= \sqrt{|\alpha|^2} \sqrt{\langle x, x\rangle}$$

$$= |\alpha| \; \|x\|$$

(4) $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x, x+y\rangle + \langle y, x+y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \cancel{\overline{\langle x, y\rangle}} + \langle y, y\rangle$$

$$= \langle x, x\rangle + \cancel{2Re(\langle x, y\rangle)} + \langle y, y\rangle$$

$$\le \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle$$

$$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

$$\therefore \|x+y\| \le \|x\| + \|y\|$$

~~note let $x \in \mathbb{C}$, $x = a+bi$, $a, b \in \mathbb{R}$~~

$$\cancel{x + \bar{x} = a+bi + a-bi}$$

$$\cancel{= a+bi + a-bi}$$

$$\cancel{= 2a}$$

$$\cancel{= 2Re(x)}$$

Example

take $x = (1-2i,\ i)$ $\quad X = \mathbb{C}^2$

$y = ~~(\ldots)~~\ (3,\ 1-2i)$

1 $\quad z = (-3i,\ 1+i)$ in Example 1

Show that $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

2 if $\alpha = 2i$, $\beta = 3$

~~find $\langle \alpha x$~~

~~Show that $\langle \alpha x + \beta y, z\rangle$~~

show that $\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$

3 find $\langle x, 0\rangle$ and $\langle 0, y\rangle$

4 find $\|x\|$, $\|y\|$, $\|z\|$

$\|x+y\|$

Solution

1 $\quad x = (1-2i,\ i)$

~~$y+z = (2-i) + (-3i$~~

~~$= 2-4i$~~

~~$\langle x, y+z\rangle = \ldots 8-2i$~~

$y+z = (3,\ 1-2i) + (-3i,\ 1+i)$

$= (3 + -3i,\ 1-2i + 1+i)$

$= (3-3i,\ 2-i)$

$\langle x, y+z\rangle = \langle (1-2i,\ i),\ (3-3i,\ 2-i)\rangle$

$= x_1\overline{(y_1+z_1)} + x_2\overline{(y_2+z_2)}$

$= (1-2i)\overline{(3-3i)} + i\,\overline{(2-i)}$

$= (1-2i)(3+3i) + i(2+i)$

$= (3 - -6) + (-6+3)i + (2i-1)$

$= 9 + -3i + (-1+2i)$

$\boxed{= 8 - i}$

~~10~~ 15

Let $x = (1-2i, i)$

$y = (3, 1-2i)$

$z = (-3i, 1+i)$ for example (1)

$= \langle (1-2i, i) / (3, 1-2i) \rangle$

$\langle x, y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$

$= (1-2i)\bar{3} + i\,\overline{(1-2i)}$

$= (1-2i)3 + i(1+2i)$

$= 3 - 6i + i - 2$

$= 1 - 5i$

$\langle x, z \rangle = x_1 \bar{z}_1 + x_2 \bar{z}_2$

$= (1-2i)\,\overline{(-3i)} + i\,\overline{(1+i)}$

$= (1-2i)3i + i(1-i)$

$= 3i + 6 + i + 1$

$= 7 + 4i$

$\langle x, y \rangle + \langle x, z \rangle = 1 - 5i + 7 + 4i$

$= \underline{8 - i}$

now for (2) $\alpha = 2i$, $\beta = 3$

$\alpha y + \beta z = 2i(3, 1-2i) + 3(-3i, 1+i)$

$= (6i, 2i+4) + (-9i, 3+3i)$

$= (6i + -9i, 4 + 2i + 3 + 3i)$

$= (-3i, 7+5i)$

$\langle x, \alpha y + \beta z \rangle = \langle (1-2i, i) / (-3i, 7+5i) \rangle$

$= x_1 \overline{(\alpha y_1 + \beta z_1)} + x_2 \overline{(\alpha y_2 + \beta z_2)}$

$= (1-2i)$

16

$= (1-2i)\overline{(-3i)} + \cancel{\ldots}\, i\, \overline{(7+5i)}$

$= (1-2i)\,3i + \cancel{\ldots}\, i(7-5i)$

$= 3i + 6 + \cancel{5}\, 7i \cancel{+} 5$

$\langle x, \alpha y + \beta z \rangle = 11 + 10i$

$\because \langle x, y \rangle = 1-5i$

So $\overline{\alpha} \langle x, y \rangle = \overline{2i}\,(1-5i)$
$= -2i\,(1-5i)$
$= -2i - 10$
$= -10 - 2i$

$\because \langle x, z \rangle = 7+4i$

$\overline{\beta} \langle x, z \rangle = \overline{3}\,(7+4i)$
$= 3(7+4i)$
$= \cancel{21 + 7i}$
$= 21 + 12i$

$\overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle = -10 - 2i + 21 + 7i$
$= (-10 + 21) + (-2 + 12) i$
$= 11 + 10i$

(3) $\mathbb{C}^2$ $0 = (0,0)$ $\in \mathbb{C}^2$

$\langle x, 0 \rangle = (1-2i)\overline{(0)} + i\,\overline{(0)}$
$= 0 \in \mathbb{C}$

$\langle 0, y \rangle = 0\,\overline{3} + 0 \cdot \overline{(1-2i)}$
$= 0$

(4) $||x|| = \sqrt{\langle x, x \rangle}$ $x = (1-2i, i)$

or $||x||^2 = \langle x, x \rangle$

$x = (1-2i, i)$

$\langle x, x \rangle = (1-2i)\overline{(1-2i)} + i\,\overline{i}$

$= (1^2 + 4^2) + 1$

$= 6$

$||x|| = \sqrt{6}$

$||y|| = \sqrt{\langle y, y \rangle}$ $y = (3, 1-2i)$

$= \langle (3, 1-2i), (3, 1-2i) \rangle$

$\langle y, y \rangle = 3\,\overline{3} + (1-2i)\overline{(1-2i)}$

$= 9 + (1^2 + 4^2)$

$= 14$

$||y|| = \sqrt{14}$

$||z|| = \sqrt{\langle z, z \rangle}$ $z = \langle -3i, 1+i)$

$= \langle$

$= -3i\,\overline{-3i} + (1+i)\overline{(1+i)}$

$= 0^2 + 3^2 + (1^2 + i^2)$

$= 9 + 2$

$= 11$

$||z|| = \sqrt{11}$

$||x+y|| = ?$ $x+y = (1-2i, i) + (3, 1-2i)$

$= (1-2i+3, i + 1-2i)$

$= (4-2i, 1-i)$

$\langle x+y, x+y \rangle = (4-2i)\overline{(4-2i)} + (1-i)\overline{(1-i)}$

$= (4^2 + 2^2) + (1^2 + i^2)$

$= (8+4) + 2 = 14$

so $||x+y|| = \sqrt{14} \leq ||x|| + ||y|| = \sqrt{6} + \sqrt{14}$

18

Lemma Schwarz inquality

Let X be an inner product space then

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|$$

for every $x, y \in X$

when the equality holds if and only if $\{x, y\}$ is a linearly dependent set.

proof: Let $x, y \in X$, $\alpha \in K$

If $y = 0$, then $\langle x, 0\rangle = 0$

$0 = |\langle x, 0\rangle| \leq \|x\|\,\|y\| = 0$

$0 \leq 0$ → it is true

now let $y \neq 0$

$0 \leq \|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y\rangle$

Jeplião

$= \langle x, x - \alpha y\rangle - \alpha\langle y, x - \alpha y\rangle$

$= \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha\langle y, x\rangle + \alpha\bar{\alpha}\langle y, y\rangle$

since $y \neq 0 \rightarrow \langle y, y\rangle \neq 0$, let $\bar{\alpha} = \dfrac{\langle y, x\rangle}{\langle y, y\rangle}$

$= \langle x, x\rangle - \dfrac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle - \alpha\langle y, x\rangle + \alpha\dfrac{\langle y, x\rangle}{\cancel{\langle y, y\rangle}}\cancel{\langle y, y\rangle}$

$0 \leq \langle x, x\rangle - \dfrac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle \cancel{- \alpha\langle y, x\rangle + \alpha\langle y, x\rangle}$

$\leq \|x\|^2 - \dfrac{\langle y, x\rangle}{\|y\|^2}\langle x, y\rangle$

$\dfrac{\langle y, x\rangle\langle x, y\rangle}{\|y\|^2} \leq \|x\|^2$

$$|\langle y, x\rangle\langle x, y\rangle \leq \|x\|^2\|y\|^2$$

$\because \quad |x| = |\bar{x}|$ for any $x \in \mathbb{C}$ or $x \in \mathbb{R}$

[illegible] $\quad \langle y, x\rangle = \overline{\langle x, y\rangle}$

$$|\langle y, x\rangle| = |\overline{\langle x, y\rangle}| = |\langle x, y\rangle|$$

So $\quad |\langle y, x\rangle\langle x, y\rangle| = |\langle y, x\rangle||\langle x, y\rangle|$

عدد مركب   عدد مركب

$$= |\langle x, y\rangle||\langle x, y\rangle|$$

$$= |\langle x, y\rangle|^2$$

$$\therefore \quad |\langle y, x\rangle\langle x, y\rangle| \leq \|x\|^2\|y\|^2$$

So $\quad |\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2$

بجذر التربيعي $\quad |\langle x, y\rangle| \leq \|x\|\,\|y\|$

for every $x, y \in K$

The equality holds if $y = 0$ or $x = 0$

and if $\|x - \alpha y\|^2 = 0$

implies that $x - \alpha y = 0 \longrightarrow x = \alpha y$

that is mean $\{x, y\}$ are linearly dependence.

~~12~~   20

~~Example~~ for example 1

if we take $x = (2-i, 1+2i)$

$y = (1-i, 3+i)$

show that $|\langle x, y\rangle| \le \|x\| \, \|y\|$

solution

$\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2}$

$= (2-i)\overline{(1-i)} + (1+2i)\overline{(3+i)}$

$= (2-i)(1+i) + (1+2i)(3-i)$

$= (2 - -1) + (-1+2)i + (3 - -2) + (6 + -1)i$

$= 3 + i + 5 + 5i$

$= 8 + 6i$

$|\langle x, y\rangle| = |8+6i| = \sqrt{8^2 + 6^2} = \sqrt{64+36} = 10$

note that ~~that~~ let $w \in \mathbb{C}$, $w = a+bi$, $a, b \in R$

$|w| = \sqrt{a^2+b^2}$

$\|x\| = \sqrt{\langle x, x\rangle}$

$\langle x, x\rangle = x_1\overline{x_1} + x_2\overline{x_2}$

$= (2-i)\overline{(2-i)} + (1+2i)\overline{(1+2i)}$

$= (2-i)(2+i) + (1+2i)(1-2i)$

$= (2^2+1^2) + (1^2+2^2)$

$= 5 + 5 = 10$

$\|x\| = \sqrt{10}$