

Advanced Calculus (2)

Masses and Moments in Three Dimensions

الكتل والعزوم في الأبعاد الثلاثية

Example 1: Find the center of mass of a solid of constant density δ bounded below by the disk (القرص) $x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid (الجسم المكافئ الدوراني) $z = 4 - x^2 - y^2$.

Solution:

$$M = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} \delta \, dz \, dy \, dx = \iint_R \int_0^{4-x^2-y^2} \delta \, dz \, dy \, dx$$
$$\iint_R \delta \, z \Big|_0^{4-x^2-y^2} \, dy \, dx = \delta \iint_R [4 - x^2 - y^2] \, dy \, dx$$

Polar Coordinates

$$\delta \iint_R [4 - x^2 - y^2] \, dy \, dx = \delta \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta$$
$$= \delta \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta = \delta \int_0^{2\pi} \left[\frac{4r^2}{2} - \frac{r^4}{4} \right]_0^2 \, d\theta = \delta \int_0^{2\pi} 8 - 4 \, d\theta$$
$$= \delta \int_0^{2\pi} 4 \, d\theta = \delta \, 4\theta \Big|_0^{2\pi} = 8\pi\delta$$

The mass is $M = 8\pi\delta$

$$M_{yz} = \iiint_R \int_0^{4-x^2-y^2} \delta \, x \, dz \, dy \, dx = \delta \iint_R xz \Big|_0^{4-x^2-y^2} \, dy \, dx$$
$$\delta \iint_R x[4 - x^2 - y^2] \, dy \, dx =$$
$$\delta \int_0^2 \int_0^{2\pi} (r \cos \theta)(4 - r^2) r \, d\theta \, dr = \delta \int_0^2 \int_0^{2\pi} (4r^2 \cos \theta - r^4 \cos \theta) \, d\theta \, dr$$
$$= \delta \int_0^2 4r^2 \sin \theta - r^4 \sin \theta \Big|_0^{2\pi} \, d\theta \, dr = \delta \int_0^2 [0] \, dr = 0$$

$$\therefore M_{yz} = 0$$

Advanced Calculus (2)

$$\begin{aligned}
 M_{xz} &= \iint_R \int_0^{4-x^2-y^2} \delta \, y \, dz \, dy \, dx = \delta \iint_R yz \Big|_0^{4-x^2-y^2} dy \, dx \\
 &= \delta \iint_R y[4-x^2-y^2] dy \, dx = \\
 &\delta \int_0^2 \int_0^{2\pi} (r \sin \theta)(4-r^2) r \, d\theta dr = \delta \int_0^2 \int_0^{2\pi} (4r^2 \sin \theta - r^4 \sin \theta) d\theta \, dr \\
 &= \delta \int_0^2 -4r^2 \cos \theta + r^4 \cos \theta \Big|_0^{2\pi} dr \\
 &\delta \int_0^2 [-4r^2 + r^4 - [-4r^2 + r^4]] dr = \delta \int_0^2 [-4r^2 + r^4 + 4r^2 - r^4] dr \\
 &= \delta \int_0^2 [0] dr = 0 \\
 \therefore M_{xz} &= 0
 \end{aligned}$$

$$\begin{aligned}
 M_{xy} &= \iint_R \int_0^{4-x^2-y^2} \delta \, z \, dz \, dy \, dx = \delta \iint_R \frac{z^2}{2} \Big|_0^{4-x^2-y^2} dy \, dx \\
 &\frac{\delta}{2} \iint_R [4-x^2-y^2]^2 dy \, dx = \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4-r^2)^2 r \, dr \, d\theta \\
 &= \frac{\delta}{2(-2)} \int_0^{2\pi} \int_0^2 (-2)(4-r^2)^2 r \, dr \, d\theta = \frac{\delta}{-4} \int_0^{2\pi} \frac{(4-r^2)^3}{3} \Big|_0^2 d\theta \\
 &\frac{\delta}{-12} \int_0^{2\pi} [(4-4)^3 - (4-0)^3] d\theta = \frac{\delta}{-12} \int_0^{2\pi} -64 d\theta = \frac{64\delta}{12} \int_0^{2\pi} d\theta \\
 &= \frac{32\delta}{6} \int_0^{2\pi} d\theta = \frac{32\delta}{6} \theta \Big|_0^{2\pi} = \frac{32\delta}{6} 2\pi = \frac{32\pi\delta}{3} \\
 \therefore M_{xy} &= \frac{32\pi\delta}{3}
 \end{aligned}$$

Center of mass is

$$\begin{aligned}
 \bar{X} &= \frac{M_{yz}}{M} = \frac{0}{8\pi\delta} = 0, & \bar{y} &= \frac{M_{xz}}{M} = \frac{0}{8\pi\delta} = 0, \\
 \bar{Z} &= \frac{M_{xy}}{M} = \frac{\frac{32\pi\delta}{3}}{8\pi\delta} = \frac{4}{3}
 \end{aligned}$$

Advanced Calculus (2)

Example 2: Find the mass in the first octant of the solid inside the cylinder $x^2 + y^2 = a^2$ and between the planes $z = 0, z = h$, and has density $\delta = x$.

Solution:

$$\begin{aligned} M &= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^h \delta \, dz \, dy \, dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^h x \, dz \, dy \, dx \\ &= \int_0^a \int_0^{\sqrt{a^2-x^2}} xz \Big|_0^h \, dy \, dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} xh \, dy \, dx = \int_0^a xh \, y \Big|_0^{\sqrt{a^2-x^2}} \, dx \\ &= \int_0^a xh(a^2 - x^2)^{\frac{1}{2}} \, dx = \frac{h}{(-2)} \int_0^a (-2)x (a^2 - x^2)^{\frac{1}{2}} \, dx = \frac{h}{(-2)} \frac{(a^2 - x^2)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^a \\ &= \frac{-h}{3} \left[(a^2 - a^2)^{\frac{3}{2}} - (a^2 - 0)^{\frac{3}{2}} \right] = \frac{-h}{3} (-a^3) = \frac{a^3 h}{3} \\ \therefore M &= \frac{a^3 h}{3} \end{aligned}$$