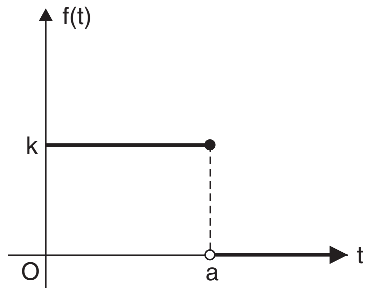
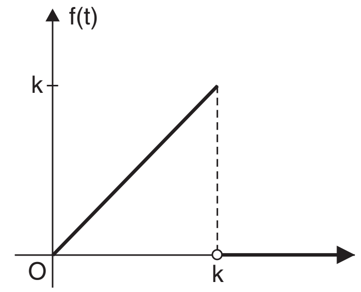
**8.** Find the Laplace transform of the functions whose graphs are given :

(*i*)




(*ii*)




**9.** Show that :

(*i*) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ (*ii*) $\int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

## Answers

**1.** (*i*) $\frac{3\sqrt{\pi}}{4s^{5/2}}, s > 0$ (*ii*) $\frac{10\,\Gamma(2/3)}{9s^{8/3}}, s > 0$ (*iii*) $\frac{5040}{s^8}, s > 0$

(*iv*) $\frac{1}{s-3}, s > 3$ (*v*) $\frac{1}{s+10}, s > -10$ (*vi*) $\frac{s}{s^2-25}, s > 5$

(*vii*) $\frac{s}{s^2-49}, s > 7$ (*viii*) $\frac{4}{s^2+16}, s > 0$ (*ix*) $\frac{s}{s^2+36}, s > 0$

**2.** $\frac{1}{s^2},\ s > 0$ **3.** $\frac{3s+4}{s^2},\ s > 0$ **4.** $\frac{s}{s^2-a^2},\ s > |a|$

**5.** $\frac{s}{s^2+a^2},\ s > 0$ **6.** (*i*) $\frac{2}{s}(2 - e^{-3s}),\ s > 0$ (*ii*) $\frac{1}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2},\ s > 0$

**7.** $\frac{1}{s^2}\left[1 - se^{-s/2} - e^{-s}\right],\ s > 0$ **8.** (*i*) $\frac{k}{s}(1 - e^{-as}),\ s > 0$

(*ii*) $\frac{1}{s^2}(1 - e^{-ks}) - \frac{k}{s}e^{-ks},\ s > 0.$

## 1.4. LINEARITY OF THE LAPLACE TRANSFORM

**Theorem.** *If f(t) and g(t) be any functions of t for t ≥ 0 whose Laplace transforms exists and a and b be any constants, then*

*L{af(t) + bg(t)} = aL{f(t)} + bL{g(t)}.*

**Proof.** We have $\mathrm{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ and $\mathrm{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt.$

Now $\mathrm{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}\,[af(t) + bg(t)]\,dt$

$$= a\int_a^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= a\mathrm{L}\{f(t)\} + b\mathrm{L}\{g(t)\}$$

$\therefore$ $\mathbf{L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}.}$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find Laplace transform of the function cosh at, $t \geq 0$ using linearity property.*

**Sol.** We have $L(\cosh at) = L\left(\dfrac{e^{at}+e^{-at}}{2}\right)$

$$= L\left(\frac{1}{2}e^{at} + \frac{1}{2}e^{-at}\right) = \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} + \frac{1}{2}\cdot\frac{1}{s-(-a)}, s > a, s > -a$$

$$= \frac{s}{s^2-a^2}, s > |a| \qquad \left(\because \ L(e^{at}) = \frac{1}{s-a}, s > a\right)$$

$\therefore$ $$\mathbf{L(\cosh at) = \frac{s}{s^2 - a^2}, s > |a|.}$$

**Example 2.** *Find the Laplace transform of the following functions of t, $t \geq 0$ :*

(*i*) *$6t + 9$* (*ii*) *$2t^2 - t + 5$* (*iii*) *$\sin^2 t$*

(*iv*) *$\sin 2t \sin 3t$* (*v*) *$\cos^3 t$* (*vi*) *$e^{4t} + e^{2t} + t^3 + \sin^2 t$.*

**Sol.** (*i*) $L(6t + 9) = 6L(t) + L(9)$ (By linearity)

$$= 6\left(\frac{1!}{s^{1+1}}\right) + \frac{1}{s}, s > 0$$

$$= \mathbf{\frac{6+9s}{s^2}, s > 0}$$

(*ii*) $L(2t^2 - t + 5) = 2L(t^2) - L(t) + L(5)$

$$= 2\cdot\left(\frac{2!}{s^3}\right) - \frac{1!}{s^2} + \frac{5}{s}, s > 0$$

$$= \mathbf{\frac{4 - s + 5s^2}{s^3}, s > 0.}$$

(*iii*) $$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = L\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)$$

$$= \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+4}\right), s > 0$$

$$= \mathbf{\frac{2}{s(s^2+4)}, s > 0.}$$

(*iv*) $$L(\sin 2t \sin 3t) = L\left(\frac{1}{2}(2\sin 2t \sin 3t)\right)$$

$$= L\left(\frac{1}{2}\cos t - \frac{1}{2}\cos 5t\right) = \frac{1}{2}L(\cos t) - \frac{1}{2}L(\cos 5t)$$

$$= \frac{1}{2} \cdot \left(\frac{s}{s^2 + 1^2}\right) - \frac{1}{2} \cdot \left(\frac{s}{s^2 + 5^2}\right) s > |1|, s > |5|$$

$$= \mathbf{\frac{12s}{(s^2 + 1)(s^2 + 25)}}, \mathbf{s > 5}.$$

$(v)$ We have $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

$$\Rightarrow \qquad \cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$$

$$\therefore \qquad \cos^3 t = \frac{1}{4} \cos 3t + \frac{3}{4} \cos t .$$

$$\therefore \qquad L(\cos^3 t) = \frac{1}{4} L(\cos 3t) + \frac{3}{4} L(\cos t) \qquad \text{(By linearity)}$$

$$= \frac{1}{4}\left(\frac{s}{s^2 + 3^2}\right) + \frac{3}{4}\left(\frac{s}{s^2 + 1^2}\right), s > 0$$

$$= \frac{s}{4}\left[\frac{1}{s^2 + 9} + \frac{3}{s^2 + 1}\right], s > 0$$

$$= \mathbf{\frac{(s^2 + 7)\, s}{(s^2 + 9)(s^2 + 1)}}, \mathbf{s > 0}.$$

$(vi)$ $L(e^{4t} + e^{2t} + t^3 + \sin^2 t)$

$$= L(e^{4t}) + L(e^{2t}) + L(t^3) + L\left(\frac{1 - \cos 2t}{2}\right) \qquad \text{(By linearity)}$$

$$= L(e^{4t}) + L(e^{2t}) + L(t^3) + \frac{1}{2} L(1) - \frac{1}{2} L(\cos 2t)$$

$$= \frac{1}{s-4} + \frac{1}{s-2} + \frac{3!}{s^{3+1}} + \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2 + 2^2}\right), s > 4, s > 2, s > 0$$

$$= \mathbf{\frac{1}{s-4} + \frac{1}{s-2} + \frac{6}{s^4} + \frac{1}{2s} - \frac{s}{2(s^2 + 4)}}, \mathbf{s > 4}.$$

**Example 3.** *Find the Laplace transform of the function $\sin(\omega t + \delta)$, $t \ge 0$.*

**Sol.** $L(\sin(\omega t + \delta)) = L(\sin \omega t \cos \delta + \cos \omega t \sin \delta)$

$$= \cos \delta \, L(\sin \omega t) + \sin \delta \, L(\cos \omega t) \qquad \text{(By linearity)}$$

$$= \cos \delta \left(\frac{\omega}{s^2 + \omega^2}\right) + \sin \delta \left(\frac{s}{s^2 + \omega^2}\right), \; s > 0$$

$$= \mathbf{\frac{1}{s^2 + \omega^2} [\omega \cos \delta + s \sin \delta]}, \; \mathbf{s > 0}.$$

**WORKING STEPS FOR SOLVING PROBLEMS**

**Step I.** Simplify the given functions and write it as a linear combination of functions with known Laplace transforms.

**Step II.** Apply linearity theorem.

**Step III.** Simplify the terms as far as possible.

## TEST YOUR KNOWLEDGE

**1.** Find the Laplace transform of the following functions of $t$, $t \geq 0$ :

(*i*) $5t + 9$ (*ii*) $t^2 + 8t - 15$

(*iii*) $t^3 + 4$ (*iv*) $t^{10} + t^4 + 2$.

**2.** Find the Laplace transform of the following functions of $t$, $t \geq 0$ :

(*i*) $e^{5t} + t^2$ (*ii*) $e^{-4t} + e^{2t}$

(*iii*) $e^{3t} + t^3$ (*iv*) $e^{-5t} + t^2 + 6t$.

**3.** Find the Laplace transform of the following functions of $t$, $t \geq 0$ :

(*i*) $\sinh 4t + \cosh 4t$ (*ii*) $t - \sinh 2t$

(*iii*) $\sin \pi t + e^{6t}$ (*iv*) $\cos (at + b)$.

**4.** Find the Laplace transform of the following functions of $t$, $t \geq 0$ :

(*i*) $\sin^2 2t$ (*ii*) $\sin^2 3t$

(*iii*) $\cos^2 4t$ (*iv*) $7 \cos^2 t$.

**5.** Find the Laplace transform of the following functions of $t$, $t \geq 0$ :

(*i*) $\sin^3 t$ (*ii*) $\sin^3 2t$

(*iii*) $\cos^3 2t + t^2$ (*iv*) $\cos^3 4t + t$.

**6.** Find the Laplace transform of the following functions of $t$, $t \geq 0$ :

(*i*) $\sin 3t \cos 2t$ (*ii*) $\cos 5t \sin 2t$

(*iii*) $\cos 4t \cos t$ (*iv*) $\sin 3t \sin 7t$.

**7.** Find the Laplace transform of the following functions of $t$ :

(*i*) $(\sin t - \cos t)^2,\ t \geq 0$ (*ii*) $1 + \sqrt{t} + \dfrac{3}{\sqrt{t}},\ t > 0$

(*iii*) $\sin at \sin bt,\ t \geq 0$ (*iv*) $\cosh at - \cos at,\ t \geq 0$.

### Answers

**1.** (*i*) $\dfrac{5}{s^2} + \dfrac{9}{s},\ s > 0$ (*ii*) $\dfrac{2}{s^3} + \dfrac{8}{s^2} - \dfrac{15}{s},\ s > 0$

(*iii*) $\dfrac{6}{s^4} + \dfrac{4}{s},\ s > 0$ (*iv*) $\dfrac{10!}{s^{11}} + \dfrac{24}{s^5} + \dfrac{2}{s},\ s > 0$

**2.** (*i*) $\dfrac{1}{s-5} + \dfrac{2}{s^3},\ s > 5$ (*ii*) $\dfrac{2(s+1)}{(s+4)(s-2)},\ s > 2$

(*iii*) $\dfrac{1}{s-3} + \dfrac{6}{s^4},\ s > 3$ (*iv*) $\dfrac{1}{s+5} + \dfrac{2}{s^3} + \dfrac{6}{s^2},\ s > 0$

**3.** (*i*) $\dfrac{1}{s-4}, s > 4$ (*ii*) $\dfrac{4+s^2}{s^2(4-s^2)}, s > 2$

(*iii*) $\dfrac{\pi}{s^2+\pi^2} + \dfrac{1}{s-6}, s > 6$ (*iv*) $\dfrac{s\cos b - a\sin b}{s^2+a^2}, s > 0$

**4.** (*i*) $\dfrac{8}{s\,(s^2+16)}, s > 0$ (*ii*) $\dfrac{18}{s\,(s^2+36)}, s > 0$

(*iii*) $\dfrac{s^2+32}{s\,(s^2+64)}, s > 0$ (*iv*) $\dfrac{7\,(s^2+2)}{s\,(s^2+4)}, s > 0$

**5.** (*i*) $\dfrac{6}{(s^2+1)\,(s^2+9)}, s > 0$ (*ii*) $\dfrac{48}{(s^2+4)\,(s^2+36)}, s > 0$

(*iii*) $\dfrac{s\,(s^2+28)}{(s^2+4)\,(s^2+36)} + \dfrac{2}{s^3}, s > 0$ (*iv*) $\dfrac{s\,(s^2+112)}{(s^2+16)\,(s^2+144)} + \dfrac{1}{s^2}, s > 0$

**6.** (*i*) $\dfrac{3s^2+15}{(s^2+1)\,(s^2+25)}, s > 0$ (*ii*) $\dfrac{2s^2-42}{(s^2+9)\,(s^2+49)}, s > 0$

(*iii*) $\dfrac{s\,(s^2+17)}{(s^2+9)\,(s^2+25)}, s > 0$ (*iv*) $\dfrac{42s}{(s^2+16)\,(s^2+100)}, s > 0$

**7.** (*i*) $\dfrac{s^2-2s+4}{s\,(s^2+4)}, s > 0$ (*ii*) $\dfrac{1}{s} + \dfrac{\sqrt{\pi}}{2s^{3/2}} + \dfrac{3\sqrt{\pi}}{s^{1/2}}, s > 0$

(*iii*) $\dfrac{2abs}{(s^2+(a-b)^2)\,(s^2+(a+b)^2)}, s > 0$ (*iv*) $\dfrac{2a^2 s}{s^4-a^4}, s > |\,a\,|.$

## 1.5. SHIFTING THEOREMS

There are two shifting theorems. In the first theorem the variable $s$ of the function $L(f(t))$ is replaced by $s - a$ and in the second theorem the variable $t$ of the function $f(t)$ is replaced by $t - a$.

## 1.6. FIRST SHIFTING THEOREM

*If $f(t)$ be a function of $t$ for $t \geq 0$ whose Laplace transform $F(s)$ exists for $s > k$ then for any constant 'a' the function $e^{at} f(t)$ has the Laplace transform $F(s - a)$ for $s > k + a$.*

**Proof.** We have $F(s) = L(f) = \int_0^\infty e^{-st} f(t)\,dt, s > k.$

Replacing $s$ by $s - a$, we get $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt, s - a > k.$

$$\Rightarrow \quad F(s-a) = \int_0^\infty e^{-st}\,(e^{at} f(t))\,dt, s > k + a$$

$\therefore$ Laplace transform of the function $e^{at} f(t)$ is $F(s - a)$ for $s > k + a$.

**Remark.** The result of above theorem can be easily remembered as follows :

**If L(f(t)) = F(s), s > k then for any a, L($e^{at}$ f(t)) = F(s – a), s > k + a.**

**Theorem. Using first shifting theorem. Show that for $t \geq 0$ :**

**1.** $\mathbf{L(e^{at}) = \dfrac{1}{s-a}, \ s > a}$

**2.** $\mathbf{L(e^{at}\, t^{b}) = \dfrac{\Gamma(b+1)}{(s-a)^{b+1}}, b > -1, \ s > a}$

**3.** $\mathbf{L(e^{at}\, t^{n}) = \dfrac{n!}{(s-a)^{n+1}}, n = 0, 1, 2, \ldots\ldots, s > a}$

**4.** $\mathbf{L(e^{at} \sinh bt) = \dfrac{b}{(s-a)^2 - b^2}, \ s > |b| + a}$

**5.** $\mathbf{L(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}, \ s > |b| + a}$

**6.** $\mathbf{L(e^{at} \sin bt) = \dfrac{b}{(s-a)^2 + b^2}, \ s > a}$

**7.** $\mathbf{L(e^{at} \cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}, \ s > a}$.

**Proof. 1.** We have $\qquad \text{L}(1) = \dfrac{1}{s}, \ s > 0$.

$\therefore$ By **first shifting theorem,**

$$\text{L}(e^{at}\,.1) = \frac{1}{s-a}, \ s > a$$

$\therefore$
$$\mathbf{L(e^{at}) = \frac{1}{s-a}, \ s > a}.$$

**2.** We have $\text{L}(t^b) = \dfrac{\Gamma(b+1)}{s^{b+1}}, b > -1, \ s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at}\, t^{b}) = \frac{\Gamma(b+1)}{(s-a)^{b+1}}, b > -1, \ s > a}.$$

**3.** We have $\text{L}(t^n) = \dfrac{n!}{s^{n+1}}, n = 0, 1, 2, \ldots\ldots; s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at}\, t^{n}) = \frac{n!}{(s-a)^{n+1}}, n = 0, 1, 2, \ldots\ldots; s > a}.$$

**4.** We have $\text{L}(\sinh bt) = \dfrac{b}{s^2 - b^2}, \ s > |b|$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}, \ s > |b| + a}.$$

**5.** We have $L(\cosh bt) = \dfrac{s}{s^2 - b^2}$, $s > |b|$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}, \quad s > |b| + a.}$$

**6.** We have $L(\sin bt) = \dfrac{b}{s^2 + b^2}$, $s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}, \quad s > a.}$$

**7.** We have $L(\cos bt) = \dfrac{s}{s^2 + b^2}$, $s > 0$.

$\therefore$ By **first shifting theorem,**

$$\mathbf{L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}, \quad s > a.}$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the Laplace transform of the following functions* :

(*i*) $t^3 e^{-3t}$, $t \geq 0$

(*ii*) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$, $t \geq 0$

(*iii*) $e^{4t} \sin 2t \cos t$, $t \geq 0$

(*iv*) $\sinh t \cos t$, $t \geq 0$

(*v*) $e^{-t}(3 \sinh 2t - 5 \cosh 2t)$.

**Sol.** (*i*) We have $L(t^3) = \dfrac{3!}{s^{3+1}} = \dfrac{6}{s^4}$, $s > 0$.

$\therefore$ By **first shifting theorem,**

$$L(t^3 e^{-3t}) = \frac{6}{(s-(-3))^4}, s > 0 + (-3) \quad = \mathbf{\frac{6}{(s+3)^4}, \quad s > -3.}$$

(*ii*) We have $L(2 \cos 5t - 3 \sin 5t)$

$$= 2L(\cos 5t) - 3L(\sin 5t) \qquad \text{(By linearity)}$$

$$= 2\left(\frac{s}{s^2 + 5^2}\right) - 3\left(\frac{5}{s^2 + 5^2}\right), \; s > 0 \quad = \frac{2s - 15}{s^2 + 25}, \; s > 0.$$

$\therefore$ By **first shifting theorem,**

$L(e^{-3t}(2 \cos 5t - 3 \sin 5t))$

$$= \frac{2(s-(-3)) - 15}{(s-(-3))^2 + 25}, \; s > 0 + (-3) \quad = \mathbf{\frac{2s - 9}{s^2 + 6s + 34}, \quad s > -3.}$$

(*iii*) $\quad L(\sin 2t \cos t) = L\left(\dfrac{1}{2}(\sin 3t + \sin t)\right)$

$$= \frac{1}{2} \text{L}(\sin 3t) + \frac{1}{2} \text{L}(\sin t) = \frac{1}{2}\left(\frac{3}{s^2 + 3^2}\right) + \frac{1}{2}\left(\frac{1}{s^2 + 1^2}\right), s > 0$$

$$= \frac{1}{2}\left[\frac{3}{s^2 + 9} + \frac{1}{s^2 + 1}\right], s > 0.$$

$\therefore$ By **first shifting theorem,**

$$\text{L}(e^{4t} \sin 2t \cos t) = \frac{1}{2}\left[\frac{3}{(s-4)^2 + 9} + \frac{1}{(s-4)^2 + 1}\right], s > 0 + 4$$

$$\mathbf{= \frac{1}{2}\left[\frac{3}{s^2 - 8s + 25} + \frac{1}{s^2 - 8s + 17}\right], s > 4.}$$

(*iv*) $\sinh t \cos t = \dfrac{e^t - e^{-t}}{2} \cos t = e^t \left(\dfrac{\cos t}{2}\right) - e^{-t} \left(\dfrac{\cos t}{2}\right).$ $\left[\because \ \sinh at = \dfrac{e^{at} - e^{-at}}{2}\right]$

Now $\text{L}\left(\dfrac{\cos t}{2}\right) = \dfrac{1}{2} \text{L}(\cos t) = \dfrac{1}{2}\left(\dfrac{s}{s^2 + 1^2}\right) = \dfrac{s}{2(s^2+1)}, \ s > 0.$

$$\therefore \quad \text{L}(\sinh t \cos t) = \text{L}\left(e^t \cdot \frac{\cos t}{2} - e^{-t} \cdot \frac{\cos t}{2}\right)$$

$$= \text{L}\left(e^t \cdot \frac{\cos t}{2}\right) - \text{L}\left(e^{-t} \cdot \frac{\cos t}{2}\right) \qquad \text{(By linearity)}$$

$$= \frac{s-1}{2((s-1)^2 + 1)} - \frac{s-(-1)}{2((s-(-1))^2 + 1)}, \ s > 0 + 1, s > 0 + (-1)$$

$$= \frac{s-1}{2(s^2 - 2s + 2)} - \frac{s+1}{2(s^2 + 2s + 2)}, \ s > 1$$

$$\mathbf{= \frac{s^2 - 2}{s^4 + 4}, \ s > 1.}$$

(*v*) We have $\text{L}(3 \sinh 2t - 5 \cosh 2t)$

$$= 3\text{L}(\sinh 2t) - 5\text{L}(\cosh 2t) = 3 \cdot \frac{2}{s^2 - 4} - 5 \cdot \frac{s}{s^2 - 4}, \ s > |2|$$

$$= \frac{6 - 5s}{s^2 - 4}, \ s > 2.$$

By **first shifting theorem,**

$$\text{L}(e^{-t} (3 \sinh 2t - 5 \cosh 2t)) = \frac{6 - 5(s+1)}{(s+1)^2 - 4}, s > 2 + (-1) \quad = \mathbf{\frac{1 - 5s}{s^2 + 2s - 3}, s > 1.}$$

**Example 2.** *If the Laplace transform of the function f(t) of t for t ≥ 0 is F(s), then show that*

$$L[(\sinh at) f(t)] = \frac{1}{2} [F(s-a) - F(s+a)].$$

*Hence evaluate L(sinh 2t sin 3t).*