Discrete Structures
Functions

Introduction

- You have already encountered function
  - $f(x,y) = (x+y)^2$
  - $f(x) = x$
  - $f(x) = \text{sqrt}(x)$
- Here we will study functions defined on <u>discrete domains</u> and <u>ranges</u>.
- We will generalize functions to <u>mappings</u>
- We may not always be able to write function as equations as above

Definition

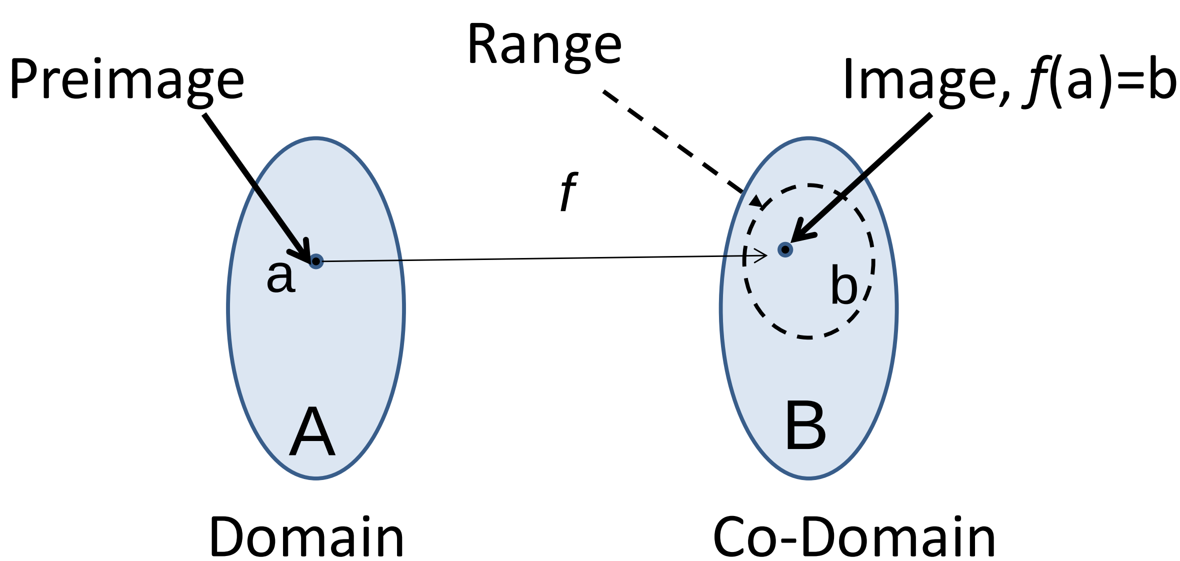
- **Definition**: A function f from a set X to a set Y is an assignment of exactly one element of B to each element of X.
- We write *f*(x)=y if b is the unique element of Y assigned by the function f to the element x∈X.
- If *f* is a function from X to Y, we write

f: X → Y

This can be read as '*f* maps X to Y'

- Note the subtlety
  - Each and every element of X has a single mapping
  - Each element of Y may be mapped to several elements in X or not at all

Function A to B



A function, *f*: A → B

Example

- Let:
  - $A = \{a_1,a_2,a_3,a_4,a_5\}$
  - $B = \{b_1,b_2,b_3,b_4,b_5\}$
  - $f=\{(a_1,b_2), (a_2,b_3), (a_3,b_3), (a_4,b_1), (a_5,b_4)\}$
  - $S=\{a_1,a_3\}$
- Draw a diagram for *f*
- What is the:
  - Domain, co-domain, range of *f*?
  - Image of S, *f*(S)?

One-to-one property

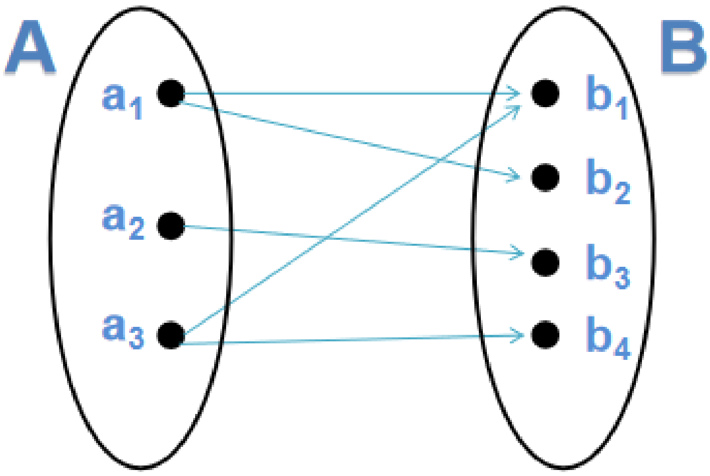
- **Definition**: A function *f* is said to be one-to-one or injective (or an injection) if

$\forall$ x and y in the domain of *f*, $f(x)=f(y) \Rightarrow x=y$

- An injection simply means that each element in the range has at most one preimage (antecedent)
- It may be useful to think of the contrapositive of this definition

$$x \neq y \;\Rightarrow\; f(x) \neq f(y)$$

Example



- The diagram above is a function or not?
- No, because each of a_1, a_3 has two images

Onto properity

- **Definition**: A function *f*: A→B is called onto or surjective (or an surjection) if

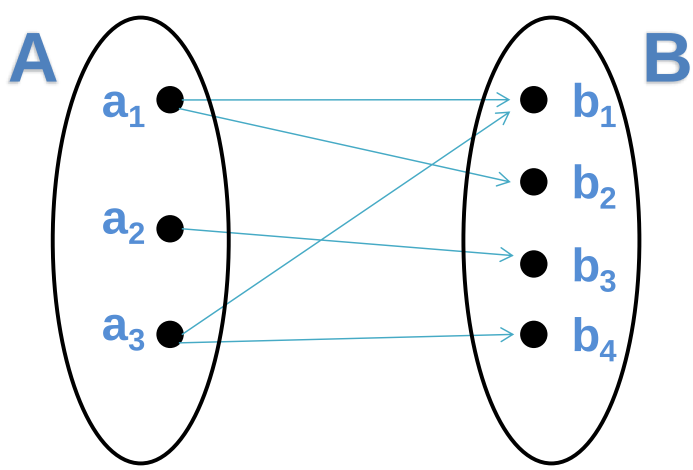
$\forall b \in B, \exists a \in A$ with $f(a)=b$

- Intuitively, a surjection means that every element in the codomain is mapped (i.e., it is an image, has an antecedent).
- Thus, the range is the same as the codomain

Bijection property

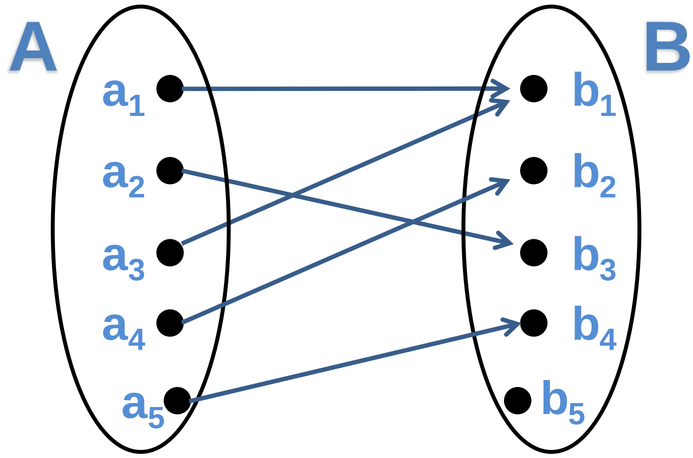
- **Definition**: A function *f* is a one-to-one correspondence (or a bijection), if is both one-to-one (injective) and onto (surjective)
- One-to-one correspondences are important because they endow a function with an <u>inverse</u>.
- They also allow us to have a concept cardinality for infinite sets
- Let's look at a few examples to develop a feel for these definitions…

Example 1



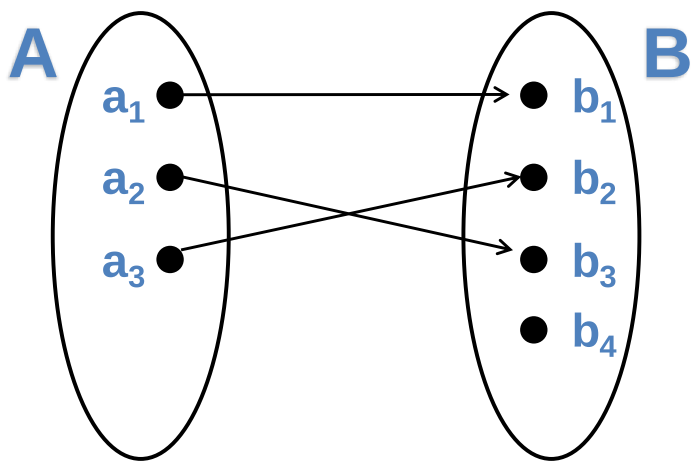
- The diagram above is a function or not?
- No, because each of a_1, a_3 has two images

Example 2



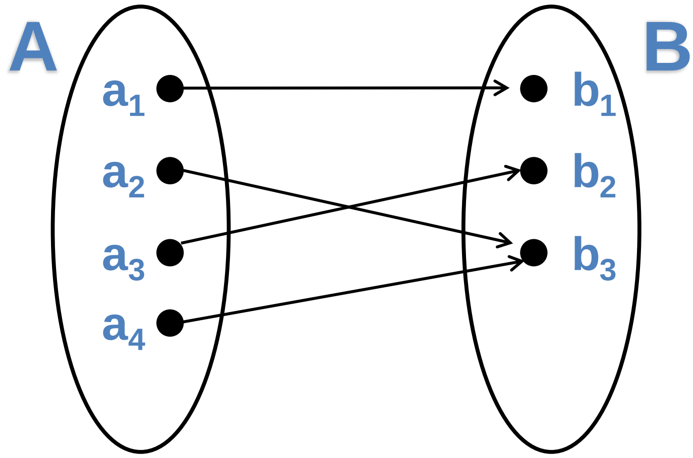
- Is this a function
  - One-to-one (injective)? Why? No, b_1 has 2 preimages
  - Onto (surjective)? Why? No, b5 has no preimage

Example 3



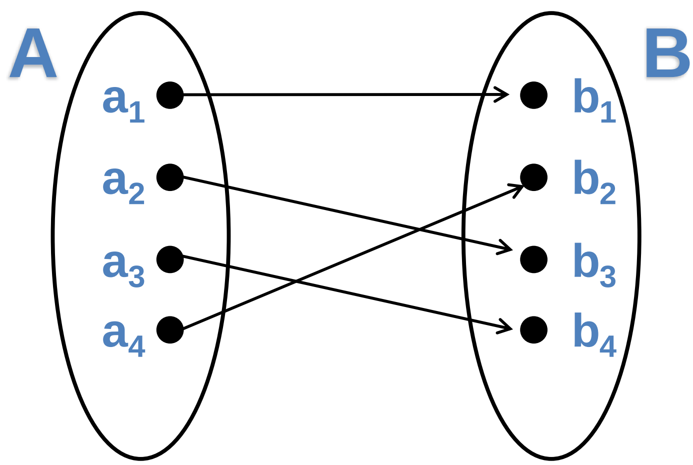
- Is this a function
  - One-to-one (injective)? Why? Yes, no b_i has 2 preimages
  - Onto (surjective)? Why? No, b_4 has no preimage

Example 4



- Is this a function
  - One-to-one (injective)? Why? No, b_3 has 2 preimages
  - Onto (surjective)? Why? Yes, every b_i has a preimage

Example 5



- Is this a function
  - One-to-one (injective)?
  - Onto (surjective)?

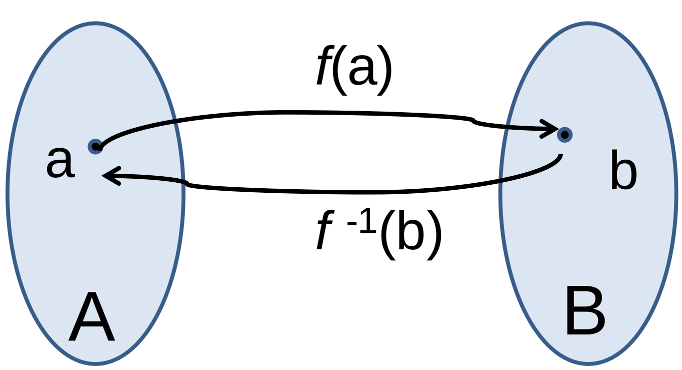
Thus, it is a bijection or a one-to-one correspondence

Inverse Functions

- **Definition**: Let f: A→B be a bijection. The <u>inverse</u> function of f is the function that assigns to an element b∈B the unique element a∈A such that f(a)=b
- The inverse function is denote f^{-1}.
- When f is a bijection, its inverse exists and

$$f(a)=b \Leftrightarrow f^{-1}(b)=a$$

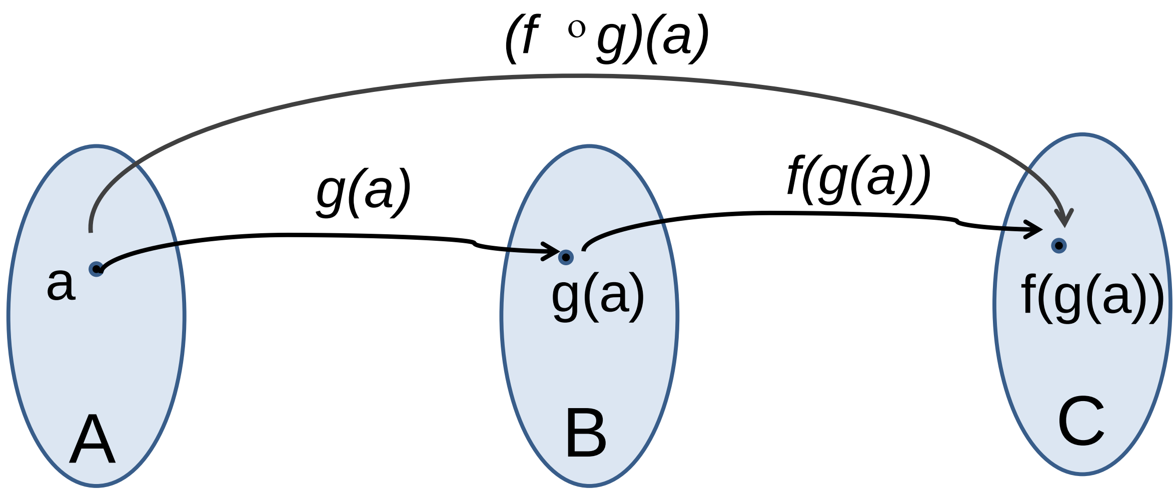
Inverse Functions: Representation



Domain Co-Domain

A function and its inverse

Composition: Graphical Representation

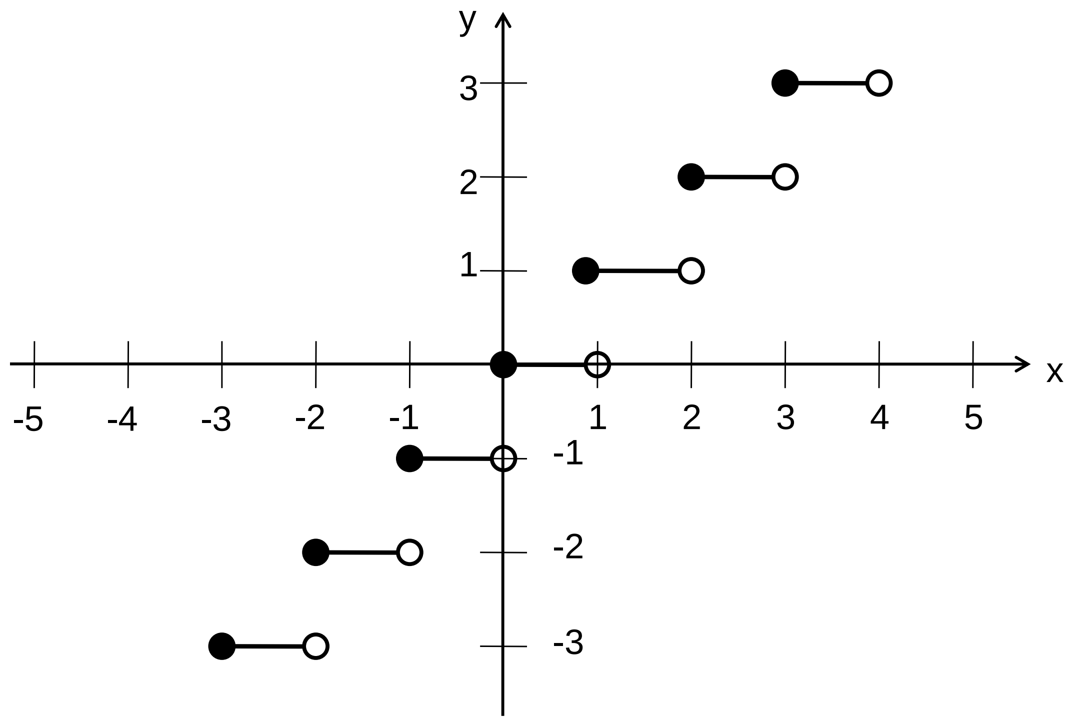


The composition of two functions

Important Functions: Floor & Ceiling

- **Definitions**:
  - The <u>floor function</u>, denoted $\lfloor x \rfloor$, is a function $\mathbb{R} \to \mathbb{Z}$. Its values is the <u>largest integer</u> that is less than or equal to x
  - The ceiling function, denoted $\lceil x \rceil$, is a function $\mathbb{R} \to \mathbb{Z}$. Its values is the <u>smallest integer</u> that is greater than or equal to x

Important Functions: Floor



Important Functions: Ceiling