Chain rule
قاعدة السلسلة

Theorem1: If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and $\frac{dz}{dt}$ is given as

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad \text{or} \quad \frac{dz}{dt} = z_x\frac{dx}{dt} + z_y\frac{dy}{dt}}$$

where the ordinary derivative are evaluated at t and the partial derivative are evaluated at (x, y).

Example 1: Use the chain rule to find $\frac{dz}{dt}$ for the function

$$z = x^2 y \quad , \qquad x = t^2 \quad , \qquad y = t$$

Solution: we must use the following rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

So,

$$\frac{\partial z}{\partial x} = 2xy \quad , \quad \frac{dx}{dt} = 2t \; , \qquad \text{and} \qquad \frac{\partial z}{\partial y} = x^2 \quad , \quad \frac{dy}{dt} = 1$$

Substitute above derivatives in rule

$$\frac{dz}{dt} = 2xy\,.\,2t + x^2.\,1 \quad \rightarrow \quad \frac{dz}{dt} = 2(t^2)(t).\,2t + (t^2)^2$$

$$\frac{dz}{dt} = 4t^4 + t^4 \qquad \rightarrow \qquad \frac{dz}{dt} = 5t^4$$

Example 2: Use the chain rule to find $\frac{dz}{d\theta}$ for the function

$$z = \sqrt{xy + y^2}, \qquad x = cos\theta \quad , \qquad y = sin\theta$$

Solution: we must use the following rule

$$\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(xy + y^2)^{-\frac{1}{2}}\, y \quad , \qquad \frac{dx}{d\theta} = -sin\theta$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(xy + y^2)^{-\frac{1}{2}}\,(x + 2y), \qquad \frac{dy}{d\theta} = cos\theta$$

$$\frac{dz}{d\theta} = \frac{1}{2}y(xy + y^2)^{-\frac{1}{2}}\,(-sin\theta) + \frac{1}{2}(x + 2y)\,(xy + y^2)^{-\frac{1}{2}}\,\cos\theta$$

$$\frac{dz}{d\theta} = \frac{-sin^2\theta}{2\sqrt{cos\theta sin\theta + sin^2\theta}} + \frac{\cos^2\theta + 2sin\theta\,\cos\theta}{2\sqrt{cos\theta sin\theta + sin^2\theta}}$$

Theorem2: (Two variable chain rule)

If $x = x(u, v)$ and $y = y(u, v)$ have first order partial derivative at the point (u, v) , and if $z = f(x, y)$ is differentiable at the point $(x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ have first order partial derivative at (u, v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

If there are three variables such as $w = f(x, y, z)$, then chain rule is

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

Example 1: Iet $z = e^{xy}$, $x = 2u + v$, $y = \frac{u}{v}$. Find $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$

Solution: To find $\frac{\partial z}{\partial u}$, we used the following formulation

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

So,

$\frac{\partial z}{\partial x} = ye^{xy}$ | $\frac{\partial z}{\partial y} = xe^{xy}$

$\frac{\partial x}{\partial u} = 2$ | $\frac{\partial y}{\partial u} = \frac{1}{v}$

$$\frac{\partial z}{\partial u} = ye^{xy}(2) + xe^{xy}\,(\frac{1}{v})$$

$$\therefore \quad \frac{\partial z}{\partial u} = 2\frac{u}{v}e^{(2u+v)(\frac{u}{v})} + \frac{1}{v}(2u + v)e^{(2u+v)(\frac{u}{v})}$$

To find $\frac{\partial z}{\partial v}$, we used the following formulation

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

So,

$\frac{\partial z}{\partial x} = ye^{xy}$ | $\frac{\partial z}{\partial y} = xe^{xy}$

$\frac{\partial x}{\partial v} = 1$ | $\frac{\partial y}{\partial v} = -\frac{u}{v^2}$

$$\frac{\partial z}{\partial u} = ye^{xy}(1) + xe^{xy}\,(-\frac{u}{v^2})$$

$$\therefore \quad \frac{\partial z}{\partial v} = \frac{u}{v}e^{(2u+v)(\frac{u}{v})} - \frac{u}{v^2}(2u + v)e^{(2u+v)(\frac{u}{v})}$$

Example 2: Iet $z = x^2 + y^2$, $x = r\,cos\theta$, $y = r\,sin\theta$. Prove that $\frac{\partial z}{\partial r} = 2r,\quad \frac{\partial z}{\partial \theta} = 0$

Solution: To find $\frac{\partial z}{\partial r}$, we used the following formulation

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial x}{\partial r} = cos\theta$

$\frac{\partial z}{\partial y} = 2y$

$\frac{\partial y}{\partial r} = sin\theta$

$\frac{\partial z}{\partial r} = 2x\ \ cos\theta + 2y\ \ sin\theta \quad \rightarrow \frac{\partial z}{\partial r} = 2(r\,cos\theta)cos\theta + 2(r\,sin\theta)sin\theta$

$\frac{\partial z}{\partial r} = 2r\,cos^2\theta + 2r\,sin^2\theta \rightarrow \frac{\partial z}{\partial r} = 2r\,[cos^2\theta + sin^2\theta]$

$$\boxed{\therefore\ \frac{\partial z}{\partial r} = 2r}$$

To find $\frac{\partial z}{\partial \theta}$, we used the following formulation

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$\frac{\partial z}{\partial x} = 2x$

$\frac{\partial x}{\partial \theta} = -r\,sin\theta$

$\frac{\partial z}{\partial y} = 2y$

$\frac{\partial y}{\partial \theta} = r\,cos\theta$

$$\frac{\partial z}{\partial \theta} = 2x(-r\,sin\theta) + 2y(r\,cos\theta)$$

$$\rightarrow \frac{\partial z}{\partial \theta} = 2r\,cos\theta(-r\,sin\theta) + 2r\,sin\theta(r\,cos\theta)$$

$$\boxed{\therefore\ \frac{\partial z}{\partial \theta} = -2r^2 sin\theta\,cos\theta + 2r^2 sin\theta\,cos\theta = 0}$$

Example 3: If $w = e^{xyz}$, $x = 3u + v$, $y = 3u - v,$ $z = u^2 v$

find $\frac{\partial w}{\partial u},$ $\frac{\partial w}{\partial v}$

Solution: To find $\frac{\partial w}{\partial u}$, we used the following formulation

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$\frac{\partial w}{\partial x} = yze^{xyz}$	$\frac{\partial w}{\partial y} = xze^{xyz}$	$\frac{\partial w}{\partial z} = xye^{xyz}$
$\frac{\partial x}{\partial u} = 3$	$\frac{\partial y}{\partial u} = 3$	$\frac{\partial z}{\partial u} = 2uv$

$$\frac{\partial w}{\partial u} = 3yze^{xyz} + 3xze^{xyz} + xye^{xyz}(2uv)$$

To find $\frac{\partial w}{\partial v}$, we used the following formulation

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$\frac{\partial w}{\partial x} = yze^{xyz}$	$\frac{\partial w}{\partial y} = xze^{xyz}$	$\frac{\partial w}{\partial z} = xye^{xyz}$
$\frac{\partial x}{\partial v} = 1$	$\frac{\partial y}{\partial v} = -1$	$\frac{\partial z}{\partial v} = u^2$

$$\frac{\partial w}{\partial v} = yze^{xyz} - xze^{xyz} + xye^{xyz}(u^2)$$

Example 4: If $z = f(x,y)$, $x = s^2 - t^2$, $y = 2st$. Prove that

$$s\ \frac{\partial z}{\partial s} - t\ \frac{\partial z}{\partial t} = 2(s^2 + t^2)\frac{\partial f}{\partial x}$$

Solution: To find $\frac{\partial z}{\partial s}$, we used the following formulation

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(2t) \quad \rightarrow \quad \boxed{\frac{\partial z}{\partial s} = 2s\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}}$$

And

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2s) \quad \rightarrow \quad \boxed{\frac{\partial z}{\partial t} = -2t\frac{\partial f}{\partial x} + 2s\frac{\partial f}{\partial y}}$$

$$s\ \left[2s\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y}\right] - t\ \left[-2t\frac{\partial f}{\partial x} + 2s\frac{\partial f}{\partial y}\right]\ =$$

$$2s^2\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y} + 2t^2\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y} = 2(s^2 + t^2)\frac{\partial f}{\partial x}$$

Example 5 : Suppose that $w = xy + yz$, $y = sinx$, $z = e^x$.
use an appropriate form (صيغة مناسبة) of the chain rule to find $\frac{dw}{dx}$.

Solution: $$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$

$$\frac{dw}{dx} = y + (x + z)cosx + ye^x$$

$$\frac{dw}{dx} = sinx\ + (x + e^x)cosx + sinxe^x$$

Example 6 : If f is a differentiable function of two variables

$z = f(x, y)$, $x = s^2 - t^2$, $y = t^2 - s^2$. show that

$t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = 0.$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}2s + \frac{\partial f}{\partial y}(-2s)$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$t\left[\frac{\partial f}{\partial x}2s + \frac{\partial f}{\partial y}(-2s)\right] + s\left[\frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)\right] = 0$$