2^{nd} Lecture

Remark (1.4): It not necessary that the union of two topologies on X form a topology on X as in the following example.

Example (1.6): Let $X = \{a, b, c\}$, and let $\tau_1 = \{\emptyset, \{a\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, X\}$.

Now $\tau_1 \cup \tau_1 = \{\emptyset, \{a\}, \{b\}, X\}$

$\tau_1 \cup \tau_1$ is not a topology on X , because $\{a\}, \{b\} \in \tau_1 \cup \tau_1$ while

$\{a\} \cup \{b\} = \{a, b\} \notin\in \tau_1 \cup \tau_1$.

Definition (1.2): Let (X, τ) be a topological space. If $E \subset X$, we say that the point $x \in X$ is a limit point of E iff

$$\forall\ open\ G \ni x\ ; (G \cap E) \{x\} \neq \emptyset.$$

Definition (1.3): The set of all limit points of E is called the derived set of E, (المشتقة) denoted by $d(E)$ or E'.

Example (1.7): Let $X = \{a, b, c, d, e\}$ and

$\tau = \{\emptyset, \{a\}, \{b, d\}, \{a, b, d\}, \{b, c, d, e\}, X\}$ is a topology on X

If $A = \{a, b, d\}$, $B = \{a, b, c\}$. Find $d(A)$.

Solution:

$a \in X$: The open sets containing a are

$\{a\}$, $\{a, b, d\}$, X

We have

$$(\{a\} \cap A) - \{a\} = (\{a\} \cap \{a, b, d\}) - \{a\}$$
$$= \{a\} - \{a\} = \emptyset$$

$\Rightarrow \quad a \notin d(A)$

$b \in X$: The open sets containing b are

$\{b, d\}$, $\{a, b, d\}$, $\{b, c, d, e\}$, X

Now $(\{b, d\} \cap \{a, b, d\}) - \{b\} = \{b, d\} \backslash \{b\} = \{d\} \neq \emptyset$

Since $\{b, d\} \subset \{b, c, d\}$ it satisfies the relation

Also $\{b, d\} \subset \{b, c, d, e\}$ hence it satisfies the relation too.

$\Rightarrow \quad b \in d(A)$

$c \in X$: The open sets containing c are

$\{b, c, d, e\}$, X

Now $(\{b, c, d, e\} \cap \{a, b, d\}) - \{c\} = \{b, d\} \backslash \{c\} = \{b, d\} \neq \emptyset$

The relation is true for $X \supset \{b, c, d, e\} \Rightarrow c \in d(A)$

$d \in X$: The open sets containing d are

$\{b, d\}$, $\{a, b, d\}$, $\{b, c, d, e\}$, X

Now $(\{b, d\} \cap A) - \{d\} = \{b, d\} - \{d\} = \{b\} \neq \emptyset$

The relation is true for the open set $\{a, b, d\}$, $\{b, c, d, e\}$, $X \supset \{b, d\}$

$\Rightarrow \quad d \in d(A)$

$c \in X$: The open sets containing e are

$\{b, c, d, e\}$, X

Now $(\{b, c, d, e\} \cap \{a, b, d\}) \backslash \{e\} = \{b, d\} - \{e\} = \{b, d\} \neq \emptyset$

The relation is true for $X \supset \{b, c, d, e\}$

$\Rightarrow e \in d(A)$

$\therefore \quad d(A) = \{b, c, d, e\}$

Remark (1.5): If one of the open set satisfies the condition of limit point (in above definition) all the open sets containing it satisfy also condition.

Example (1.8): Let (X, τ) be the weak (indiscrete) topological space and $E \subset X$. Find $d(E)$.

Solution:

We have $\tau = \{\emptyset, X\}$

Let $x \in X$ be any point

We have only open set containing x is X

Now $(X \cap E)\{x\} \neq \emptyset$ always except $E = \emptyset \cup E = \{x\}$

$$d(E) = \begin{cases} \emptyset & if \quad E = \emptyset \\ X - \{x\} & if \quad E = \{x\} \\ X & if \quad E = \{x, y\} \end{cases}$$

Theorem (1.2): Let A, B be subsets of a topological space (X, τ) then

(i) $d(\emptyset) = \emptyset$.

(ii) $A \subset B \Rightarrow d(A) \subset d(B)$.

(iii) $x \in d(E) \Rightarrow x \in d(E - \{x\})$.

(iv) $d(A \cup B) = d(A) \cup d(B)$.

Proof:

(i) Since $\forall\, x \in X$, $\forall\, open\ G \ni x\,; (G \cap \emptyset)\backslash\{x\} = \emptyset$

$\Rightarrow \; x \notin d(\emptyset)$, $\forall\, x \in X$

$\Rightarrow \; d(\emptyset) = \emptyset$

(ii) We need to show that $d(A) \subset d(B)$

Let $x \in d(A) \Rightarrow x$ is a limit point of A

$\Rightarrow \; \forall\, open\ G \ni x$, $(G \cap A)\backslash\{x\} \neq \emptyset$

Since $A \subset B$

$\Rightarrow \; x$ is a limit point of B

$\Rightarrow \; x \in d(B)$

Hence $d(A) \subset d(B)$.

(iii) Since

$$\big((G\cap E) - \{x\}\big) - \{x\} = ((G\cap E)\cap\{x\}^c)\cap\{x\}^c$$

$$= (G\cap E)\cap(\{x\}^c\cap\{x\}^c)$$

$$= (G\cap E)\cap\{x\}^c$$

$$= (G\cap E)\backslash\{x\}$$

Thus if $x \in d(E) \Rightarrow x \in d(E - \{x\})$

(iv) Since $\left.\begin{matrix} A \subset A\cup B \\ B \subset A\cup B \end{matrix}\right\} \Rightarrow \left.\begin{matrix} d(A) \subset d(A\cup B) \\ d(B) \subset d(A\cup B) \end{matrix}\right\}$

$\Rightarrow\ d(A)\cup d(B) \subset d(A\cup B)$ ……………… (1)

Now we have to prove that $d(A)\cup d(B) \subset d(A\cup B)$

Let $x \notin d(A)\cup d(B)$

$\Rightarrow\ x \notin d(A) \land x \notin d(B)$

$\Rightarrow\ x$ is not a limit point of A and x is not a limit point of B

$\Rightarrow\ \exists\ open\ G_1 \ni x\ ;\ (G_1\cap A)\backslash\{x\} = \emptyset \land \exists\ open\ G_2 \ni x\ ;\ (G_2\cap B)\backslash\{x\} = \emptyset$

Put $G = G_1\cap G_2$

$\Rightarrow\ \exists\ open\ G \ni x\ ;\ (G\cap A)\backslash\{x\} = \emptyset \land \exists\ open\ G \ni x\ ;\ (G\cap B)\backslash\{x\} = \emptyset$

$\Rightarrow\ \exists\ open\ G \ni x\ ;\ [(G\cap A) - \{x\}]\cup[(G\cap B) - \{x\}] = \emptyset$

$\Rightarrow\ \exists\ open\ G \ni x\ ;\ [G\cap(A\cup B)] - \{x\} = \emptyset$

$\Rightarrow\ x \notin d(A\cup B)$

$\Rightarrow\ d(A)\cup d(B) \subset d(A\cup B)$ …………… (2)

From (1) and (2) we get

$d(A\cup B) = d(A)\cup d(B)$
