Lemma

if M is a basis of a vector space $\bar{X}$
Then any element $x \in \bar{X}$ can be uniquely written in the form

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

where $\alpha_i \in K$
$e_i \in M$

proof: let $x \in \bar{X}$, since M is a basis
so we can write x as linear combination of element M. so there exist $\alpha_i \in K$, $1 \le i \le n$
such that

$$x = \sum_{i=1}^{n} \alpha_i e_i \quad \text{——} \quad (1) \qquad \text{where } e_i \in M$$

أخر تمثيل

Let there is another representation to x, that is mean, $\exists$ exist $\beta_i \in K$, $1 \le i \le n$ such that

$$x = \sum_{i=1}^{n} \beta_i e_i \quad \text{——} \quad (2)$$

from (1) and (2) we get

$$\sum_{i=1}^{n} \alpha_i e_i = \sum_{i=1}^{n} \beta_i e_i$$

$$\sum_{i=1}^{n} \alpha_i e_i - \sum_{i=1}^{n} \beta_i e_i = 0$$

$$\sum_{i=1}^{n} (\alpha_i - \beta_i) e_i = 0$$

$\because$ M is linearly independent
so $\alpha_i - \beta_i = 0$
$\Rightarrow \alpha_i = \beta_i$

So x is write uniquely of M.

Subspace الفضاء الجزئي

Definition

A non empty subset M of a vector space X over K is said to be subspace if M is a vector space over K.

Example

let $X = R^3$ is a vector space over R and

1- $M_1 = \{(x,y,0) : x,y \in R\}$

2- $M_2 = \{(x,0,y) : x,y \in R\}$

3- $M_3 = \{(0,x,y) : x,y \in R\}$

4 $M_4 = \{(x,y,z),\ x,y,z \in R,\ x \geq 0\}$

each of the above set represent a subspace of R^3.

Theorem

A non empty subset M of a vector space $\bar{X}$ over a field K is a subspace of $\bar{X}$ if and only if

1. $x+y \in M$ for all $x,y \in M$
2. $\alpha \cdot x \in M$ for all $x \in M,\ \alpha \in R$.

Note For any vector subspace X, there exists two subspace are $\{0\}$ zero subspace and X which are called trivaial subspace.

If M is a subspace of X which is not trivial it is called proper subspace.

Example R^3 over R is vector space

$M = \{(x, y, 0) : x, y \in R\}$ is a subspace of R^3

Solution:

Let $x, y \in M$, $x = (x_1, x_2, 0)$ $x_1, x_2 \in R$

$y = (y_1, y_2, 0)$ $y_1, y_2 \in R$

$$x + y = (x_1, x_2, 0) + (y_1, y_2, 0)$$
$$= (x_1 + y_1, x_2 + y_2, 0)$$

$\because x_1, y_1 \in R \rightarrow x_1 + y_1 \in R$

also $x_2, y_2 \in R \rightarrow x_2 + y_2 \in R$

$\therefore x + y \in M$

Let $\alpha \in K$

$$\alpha \cdot x = \alpha(x_1, x_2, 0) = (\alpha x_1, \alpha x_2, 0)$$

$\because \alpha, x_1 \in R \rightarrow \alpha x_1 \in R$

$\because \alpha, x_2 \in R \rightarrow \alpha x_2 \in R$

$\therefore \alpha \cdot x \in M$ $\forall \alpha \in K$

so M is a subspace of R^3

Example in R^3 over R

$M = \{ (0, x, y), x, y \in R, y \geq 0 \}$

M is not subspace

Solution

Let $x, y \in M$, $x = (0, x_1, x_2)$ $x_1, x_2 \in R$ $x_2 \geq 0$

$y = (0, y_1, y_2)$ $y_1, y_2 \in R$ $y_2 \geq 0$

$x + y = (0, x_1, x_2) + (0, y_1, y_2)$

$= (0, x_1 + y_1, x_2 + y_2)$

$\because x_1, y_1 \in R \longrightarrow x_1 + y_1 \in R$

$\because x_2, y_2 \in R \longrightarrow x_2 + y_2 \in R$

and $x_2 \geq 0, y_2 \geq 0$ so $x_2 + y_2 \geq 0$

$\therefore x + y \in M$

$\alpha X = \alpha (0, x_1, x_2) = (0, \alpha x_1, \alpha x_2)$

$\because \alpha, x_1 \in R, \longrightarrow \alpha x_1 \in R$

$\alpha, x_2 \in R \longrightarrow \alpha x_2 \in R$

when $\alpha \geq 0$ $x_2 \geq 0 \longrightarrow \alpha \cdot x_2 \geq 0$

but $\alpha < 0$ $x_2 \geq 0 \longrightarrow \alpha x_2 < 0$

So $\alpha X \in M$ when $\alpha < 0$

so M is not subspace

Theorem

Let M_1, M_2 be a subspace of vector space X over field K. Then

① $M_1 \cap M_2$ is a subspace of X over K

② $M_1 + M_2$ is a subspace of X over K.

proof. ①

Let $x, y \in M_1 \cap M_2$, ~~so x, y ∈ M1~~

$\because x, y \in M_1 \longrightarrow x + y \in M_1$ since M_1 is subspace

$\because x, y \in M_2 \longrightarrow x + y \in M_2$ since M_2 is subspace

$\therefore x + y \in M_1$ and M_2

So $x + y \in M_1 \cap M_2$

now let $\alpha \in K$, ~~αx ∈ M1 ∩ M2~~

$\because \alpha \in K, x \in M_1 \longrightarrow \alpha x \in M_1$ since M_1 is subspace

$\because \alpha \in K, x \in M_2 \longrightarrow \alpha x \in M_2$ since M_2 is subspace

$\therefore \alpha x \in M_1$ and M_2

So $\alpha x \in M_1 \cap M_2$

$M_1 \cap M_2$ is a subspace of X over K

proof (2) Let $x, y \in M_1 + M_2$

So either $x, y \in M_1 \longrightarrow x+y \in M_1$

since M_1 is a subspace of X

or $x, y \in M_2 \longrightarrow x+y \in M_2$

since M_2 is a subspace of X

or $x \in M_1$ and $y \in M_2 \longrightarrow x+y \in M_1 + M_2$

~~or~~ similarly if $y \in M_1$ and $x \in M_2$

Let $\alpha \in K$

either $x \in M_1 \longrightarrow \alpha x \in M_1$

since M_1 is a subspace of X

or

$x \in M_2 \longrightarrow \alpha x \in M_2$

since M_2 is a subspace of X

So $M_1 + M_2$ is a subspace of $M_1 + M_2$

Theorem

Let M_1, M_2 be a subspace of a vector space X over a field K, Then

$$\dim(M_1 + M_2) = \dim(M_1) + \dim(M_2) - \dim(M_1 \cap M_2)$$

Note The union of M_1 and M_2, $M_1 \cup M_2$ need not to be a vector space.

Example Let R^3 be a vector space over R

$M_1 = \{ (x, y, 0), x, y \in R \}$

$M_2 = \{ (0, y, z), y, z \in R \}$

Find $M_1 + M_2$, $M_1 \cap M_2$, and the the dimension of all subspace, show that $M_1 \cup M_2$ is not subspace.

Solution

It is clearly to show that M_1, M_2 are subspace of R^3

$M_1 \cap M_2 = \{ (0, y, 0), y \in R \}$

$M_1 + M_2 = \{ (x, y, z), x, y, z \in R \} = R^3$

now we find $\dim(M_1)$

Let $A_1 = \{ (1, 0, 0), (0, 1, 0) \}$ be a subset of M_1

Let $x \in M_1$, then $x = (x_1, x_2, 0), x_1, x_2 \in R \}$

$x = x_1 (1, 0, 0) + x_2 (0, 1, 0)$

$= (x_1, 0, 0) + (0, x_2, 0)$

$= (x_1, x_2, 0) \qquad \forall x \in M_1$

So $\text{span}(A_1) = M_1$

Let $\alpha_1, \alpha_2 \in R$, then

$\alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) = (0, 0, 0)$

$(\alpha_1, 0, 0) + (0, \alpha_2, 0) = (0, 0, 0)$

$(\alpha_1, \alpha_2, 0) = (0, 0, 0)$

$\alpha_1 = 0$

$\alpha_2 = 0$ linearly independent

So A_1 is a basis for M_1, so

$\dim(M_1) = 2$

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for finding $\dim(M_2)$

Let $A_2 = \{(0,1,0),(0,0,1)\}$

we to show that A_2 is a basis for M_2

Let $x \in M_2$, $x = (0, x_1, x_2)$

$x = x_1(0,1,0) + x_2(0,0,1)$

$= (0, x_1, 0) + (0, 0, x_2)$

$= (0, x_1, x_2)$.

$\forall x \in M$, then $\operatorname{span} A_2 = M_2$

Let $\alpha_1, \alpha_2 \in R$

$\alpha_1(0,1,0) + \alpha_2(0,0,1) = (0,0,0)$

$(0,\alpha_1,0) + (0,0,\alpha_2) = (0,0,0)$

$(0,\alpha_1,\alpha_2) = (0,0,0)$

$\alpha_1 = 0$

$\alpha_2 = 0$

so A_2 is linearly independent

$\therefore A_2$ is a basis for M_2

$\dim(M_2) = 2$

عدد عناصر القاعدة A_2 يمثل البعد

$M_1 \cap M_2 = \{(0, y, 0), y \in R\}$

$A_3 = \{(0, 1, 0)\}$

Let $w \in M_1 \cap M_2$ $\quad w = (0, y, 0)$

$w = y(0, 1, 0) = (0, y, 0)$

$\forall y \in M_1 \cap M_2$

$\therefore$ span $A_3 = M_1 \cap M_2$

$\alpha(0, 1, 0) = (0, 0, 0)$

$(0, \alpha, 0) = (0, 0, 0)$

$\alpha = 0$

$\therefore A_3$ is linearly independent, so

A_3 is a basis

number of element A_3 is 1, so

$\dim(M_1 \cap M_2) = 1$

$\dim(M_1 + M_2) = \dim(R^3) = 3$

So

$\dim(M_1 + M_2) = \dim(M_1) + \dim(M_2) - \dim(M_1 \cap M_2)$

$3 = 2 + 2 - 1$

$3 = 3$

Theorem

Let X be an n-dimensional vector space. Then any proper subspace M of X has dimension less than n.

proof.

If $n=0$, then $\dim X = 0$ implies that $X = \{0\}$ so there is no proper subspace of X. when $n \neq 0$ ~~when~~ we have $\dim X \geq 1$ and may be have proper subspace.

note that if $\dim(M) = 0$ implies that $M = \{0\}$ which is not proper sub space.

So when $M \neq X$, ~~the number of element~~ ~~of~~ the basis which generated M can not generated X, so the number of it element M it will be less than the number of the basis X

So $\dim(M) \leq \dim(X) = n$

So $\dim(M) \leq n$

if $\dim(M) = n$ implie that $M = X$ which is not proper subspace

So $\dim(M) < n$.

disjoint الانفصال الانفصال

Let M_1, M_2 are two vector subspace of a vector space X over a field K we say that M_1, M_2 are disjoint if $M_1 \cap M_2 = \{0\}$

direct summand

~~...~~ A vector space $\bar{X}$ over a field K is called direct summand if there exists two vector space M_1 and M_2 such that

1. $\bar{X} = M_1 + M_2$
2. $M_1 \cap M_2 = \{0\}$ are disjoint

we write $\bar{X} = M_1 \oplus M_2$ if satisfies 1 & 2 for direct summand.

Example Let R^3 be a vector space over R such that

$M_1 = \{(0, y, z), y, z \in R\}$

$M_2 = \{(x, 0, 0), x \in R\}$

$M_3 = \{(x, y, 0), x, y \in R\}$

find $M_1 \cap M_2$, $M_1 \cap M_3$; $M_2 \cap M_3$

is R^3 has direct summand

Solution

$M_1 \cap M_2 = \{(0,0,0)\}$ disjoint

$\dim(M_1 \cap M_2) = 0$

$M_1 \cap M_3 = \{(0,y,0)\}$

$\dim(M_1 \cap M_3) = 1$

$M_2 \cap M_3 = \{(x,0,0)\}$

$\dim(M_2 \cap M_3) = 1$

now $M_1 + M_2 = \{(x,y,z), x,y,z \in R\} = R^3$

$M_1 \cap M_2 = \{0\}$

so $R^3 = M_1 \oplus M_2$ direct summand

$M_1 + M_3 = \{x,y,z), x,y,z \in R\} = R^3$

$M_1 \cap M_3 \neq \{0\}$

so R^3 is not direct summand of M_1 & M_3

$M_2 + M_3 = \{(x,y,0), x,y \in R\} \subseteq R^3$

$M_2 + M_3$ is a proper subspace of R^3 not equal.

Theorem

Let M_1 and M_2 are two subspace of a vector space $\bar{X}$ over field K. Then $X = M_1 \oplus M_2$ if and only if for any $x \in \bar{X}$ can be written as a unique representation $x = m_1 + m_2$, $m_1 \in M_1$, $m_2 \in M_2$

proof:

Suppose that $X = M_1 \oplus M_2$, So for any $x \in \bar{X}$, there exists $m_1 \in M_1$, $m_2 \in M_2$ such that $x = m_1 + m_2$, let x has another representation $x = y_1 + y_2$ for $y_1 \in M_1$, $y_2 \in M_2$

so $y_1 + y_2 = m_1 + m_2$

$$y_1 - m_1 = m_2 - y_2$$

$\because y_1, m_1 \in M_1 \rightarrow y_1 - m_1 \in M_1$

also $m_2 - y_2 \in M_1$

$\because y_2, m_2 \in M_2 \rightarrow y_2 - m_2 \in M_2$

now $m_2 - y_2 \in M_1 \cap M_2 = \{0\}$

so $m_2 - y_2 = 0 \rightarrow m_2 = y_2$

similary we get that $y_1 = m_1$

So x is uniquely written by M_1 And M_2.

Conversly

Let $x = m_1 + m_2$, for $m_1 \in M_1$, $m_2 \in M_2$

with unique representation

$\therefore$ $x = m_1 + m_2$ for $m_1 \in M_1$ and $m_2 \in M_2$

then $X = M_1 + M_2$

now let $x \in M_1 \cap M_2$, Then $x = m_1 + m_2$

has a unique representation

if $x = 0$, then $x = 0 + 0$

if not, that is mean $x \neq 0$, so there exists

$m_1 \in M_1$ and $m_2 \in M_1$ such that

$x = m_1 + m_2$

$m_2 = x - m_1 \in M_1$ which is contradication

with unique representation so must $x = \{0\}$

implies that $M_1 \cap M_2 = \{0\}$

so $X = M_1 \oplus M_2$