Examples (2)

Euclidean space R^n over R, $n \in N$. The space R^n is a Hilbert space with inner product defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

when $x = (x_1, x_2, \ldots, x_n)$
$y = (y_1, y_2, \ldots, y_n)$

proof: to show that $\langle \cdot, \cdot \rangle$ is inner product on R^n

IP_1 ~~$\langle x+y, z \rangle$~~ Let $z = (z_1, z_2, \ldots, z_n)$

~~$\langle x+y, z \rangle =$~~ $x+y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

$$\begin{aligned}
\langle x+y, z \rangle &= (x_1+y_1) \cdot z_1 + (x_2+y_2) z_2 + \cdots + (x_n+y_n) z_n \\
&= x_1 z_1 + y_1 z_1 + x_2 z_2 + y_2 z_2 + \cdots + (x_n z_n + y_n z_n \\
&= x_1 z_1 + x_2 z_2 + \cdots + x_n z_n + y_1 z_1 + y_2 + z_2 + \cdots + y_n z_n \\
&= \langle x, z \rangle + \langle y, z \rangle
\end{aligned}$$

Let $\alpha \in R$

~~$\langle \alpha x, z \rangle =$~~ $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$

$$\begin{aligned}
\langle \alpha x, y \rangle &= (\alpha x_1) y_1 + (\alpha x_2) y_2 + \cdots + (\alpha x_n) y_n \\
&= \alpha x_1 y_1 + \alpha x_2 y_2 + \cdots + \alpha x_n y_n \\
&= \alpha (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n) \\
&= \alpha \langle x, y \rangle
\end{aligned}$$

$IP_2)$ $\langle x, x\rangle = x_1x_1 + x_2x_2 + \cdots\ x_nx_n$

$= x_1^2 + x_2^2 + \cdots x_n^2$

$\because x_i^2 \geq 0 \quad \forall\, 1 \leq i \leq n,\ n \in \mathbb{N}$

so $\langle x, x\rangle = \sum_{i=1}^{n} x_i^2 \geq 0$

$\langle x, x\rangle = 0 \iff x_1x_1 + x_2x_2 + \cdots + x_nx_n = 0$

$\iff x_1^2 + x_2^2 + \quad + x_n^2 = 0$

$\iff$ so $x_i^2 = 0 \quad 1 \leq i \leq n$

$\iff \therefore x_i = 0 \quad 1 \leq i \leq n$

$\iff x = (0, 0, \cdots, 0)$

$IP_3)$ $\overline{\langle y, x\rangle} = \overline{y_1x_1 + y_2x_2 + \cdots\ y_nx_n}$

$= \overline{y_1x_1} + \overline{y_2x_2} + \cdots \overline{y_nx_n}$

$\because y_i, x_i \in R$ so $\overline{y_ix_i} = \cancel{x_iy_i}\ \overline{x_i}\,\overline{y_i} = x_iy_i \quad 1 \leq i \leq n$

$= x_1y_1 + x_2y_2 + \cdots + x_ny_n$

$= \langle x, y\rangle$

$\therefore R^n$ with $\langle\cdot,\cdot\rangle$ is inner product space (pre Hilbert space)

since R^n is complete, then R^n is ~~Banach~~ Hilbert space.

$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$

euclidean norm.

Example (3)
Unitary space $\mathbb{C}^n$, $n \in \mathbb{N}$ over $\mathbb{C}$
is a Hilbert space with inner product given by.

$$\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \dots \; x_n \overline{y_n}$$

where $x = (x_1, x_2, — x_n)$
$y = (y_1, y_2, — y_n)$ complex
and $x_i, y_i \in \mathbb{C}$ $1 \le i \le n$, $n \in \mathbb{N}$

Solution ~~At first we~~ to show that $\langle \cdot, \cdot \rangle$ is an inner product space (pre Hilbert space) on $\mathbb{C}^n$

$IP_1)$ $z = (z_1, z_2, — z_n)$ $z_i \in \mathbb{C}$ $1 \le i \le n$, $n \in \mathbb{N}$

$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$$\langle x+y, z \rangle = (x_1+y_1)\overline{z_1} + (x_2+y_2)\overline{z_2} + \dots (x_n+y_n)\overline{z_n}$$
$$= x_1\overline{z_1} + y_1\overline{z_1} + x_2\overline{z_2} + y_2\overline{z_2} + \dots \; x_n\overline{z_n} + y_n\overline{z_n}$$
$$= x_1\overline{z_1} + x_2\overline{z_2} + \dots + x_n\overline{z_n} + y_1\overline{z_1} + y_2\overline{z_2} + \dots y_n\overline{z_n}$$
$$= \langle x, z \rangle + \langle y, z \rangle$$

Let $\alpha \in \mathbb{C}$ $\alpha x = (\alpha x_1, \alpha x_2, \dots \alpha x_n)$

$$\langle \alpha x, y \rangle = (\alpha x_1)\overline{y_1} + (\alpha x_2)\overline{y_2} + \dots — (\alpha x_n)\overline{y_n}$$
$$= \alpha x_1 \overline{y_1} + \alpha x_2 \overline{y_2} + \dots — \alpha x_n \overline{y_n}$$
$$= \alpha (x_1\overline{y_1} + x_2\overline{y_2} + \dots \; x_n\overline{y_n})$$
$$=$$

$IP_2)$ $\langle x, x \rangle = x_1\overline{x_1} + x_2\overline{x_2} + \dots x_n\overline{x_n}$

~~for $x_i \in \mathbb{C}$ then $\bar{x}_i$~~

Note

Let $w \in \mathbb{C}$ such that $w = a+bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, so $\overline{w} = a-bi$, now

$$w \cdot \overline{w} = (a+bi)\cdot(a-bi)$$
$$= (a^2+b^2) + \underbrace{(ab-ba)}_{=0}i$$
$$w \cdot \overline{w} = a^2+b^2$$
$$|w| = \sqrt{a^2+b^2} = \sqrt{w \cdot \overline{w}}$$
$$\therefore |w|^2 = w \cdot \overline{w}$$

Complex number

$$\langle x, x \rangle = x_1\overline{x_1} + x_2\overline{x_2} + \dots x_n\overline{x_n}$$
$$= |x_1|^2 + |x_2|^2 + \dots |x_n|^2$$
$$\because |x_i|^2 \geq 0 \text{ for } 1 \leq i \leq n, \; n \in \mathbb{N}$$
$$\therefore |x_i| \geq 0 \quad 1 \leq i \leq n, \; n \in \mathbb{N}$$

so $\langle x, x \rangle = \sum_{i=1}^{n} |x_i|^2 \geq 0$.

now let $\langle x, x \rangle = 0 \iff x_1\overline{x_1} + x_2\overline{x_2} + \dots x_n\overline{x_n} = 0$

$$\iff |x_1|^2 + |x_2|^2 + \dots |x_n|^2 = 0$$
$$\iff |x_i|^2 = 0 \;\; \forall i \;\; 1 \leq i \leq n, \; n \in \mathbb{N}$$
$$\iff x_i = 0 \;\; \forall i \;\; 1 \leq i \leq n$$
$$\iff x = (0, 0, \dots, 0)$$

$IP_3)$ $\overline{\langle y, x\rangle} = \overline{y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots\ y_n\bar{x}_n}$

$$= \overline{y_1\bar{x}_1} + \overline{y_2\bar{x}_2} + \cdots\ \overline{y_n\bar{x}_n}$$

$$= \bar{\bar{x}}_1\bar{y}_1 + \bar{\bar{x}}_2\bar{y}_2 + \cdots\ \bar{\bar{x}}_n\bar{y}_n$$

$$= x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots\ x_n\bar{y}_n$$

$$= \langle x, y\rangle$$

$\therefore$ $\mathbb{C}^n$ over $\mathbb{C}$ with $\langle\cdot,\cdot\rangle$ is an inner product space, since $\mathbb{C}^n$ is complete, then $\mathbb{C}^n$ is Hilbert space with norm

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{|x_1|^2 + |x_2|^2 + \cdots |x_n|^2}$$

$$= \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

~~Example let $C[0,1]$ the space of all continuous functions on the interval $[0,1]$ then the inner product on it given for $x(t), y(t) \in C[0,1]$~~

Example let X be the vector space of all continuous complex-valued function on real interval $[a,b]$, $a,b \in \mathbb{R}$, so $x \in X$

$$x : [a,b] \longrightarrow \mathbb{C}$$
$$t \longrightarrow x(t) \in \mathbb{C}$$

has domain real interval $[a,b]$ and codomain is the complex field $\mathbb{C}$.

Example (1) Let $x(\cancel{t}), y(\cancel{t}) \in X$, then the inner product given by

$$\langle x, y \rangle = \int_a^b x(t)\,\overline{y(t)}\,dt$$

for example $x(t) = t$

$y(t) = 1 + ti$

So

$$\langle x / y \rangle = \int_a^b x(t)\,\overline{y(t)}\,dt$$

$$= \int_a^b t\,\overline{1+it}\,dt$$

$$= \int_a^b t\,(1 - ti)\,dt$$

$$= \int_a^b (t - t^2 i)\,dt$$

$$= \left.\frac{t^2}{2} - \frac{t^3}{3} i\,\right]_a^b$$

$$= \left(\frac{b^2}{2} - \frac{b^3}{3} i\right) - \left(\frac{a^2}{2} - \frac{a^3}{3} i\right)$$

$$= \left(\frac{b^2}{2} - \frac{a^2}{2}\right) + \left(\frac{a^3}{3} - \frac{b^3}{3}\right) i$$

where $a, b \in R$

~~The inner po.~~ So the space $\underline{X}$ with inner product is Hilbert space.

IP$_1$) Let $x, y, z \in \underline{X}$

$$\langle x+y, z\rangle = \int_a^b (x+y)(t)\,\overline{z(t)}\,dt$$

$$= \int_a^b \left(x(t)+y(t)\right] \overline{z(t)}\,dt$$

$$= \int_a^b \left(x(t)\overline{z(t)} + y(t)\overline{z(t)}\right) dt$$

$$= \int_a^b x(t)\overline{z(t)} + \int_a^b y(t)\overline{z(t)}\,dt$$

$$= \langle x, z\rangle + \langle y, z\rangle$$

Let $\alpha \in \mathbb{C}$

$$\langle \alpha x, y\rangle = \int_a^b (\alpha x)(t)\,\overline{\cancel{z}y(t)}\,dt$$

$$= \int_a^b \alpha x(t)\,\overline{y(t)}\,dt$$

$$= \alpha \int_a^b x(t)\,\overline{y(t)}\,dt$$

$$= \alpha \langle x, y\rangle$$

IP$_2$) $\langle x, x\rangle = \int_a^b x(t)\,\overline{x(t)}\,dt$

$\because\ x(t)\cdot\overline{x(t)} = |x(t)|^2$

$$= \int_a^b |x(t)|^2\,dt$$

$$\geq \left|\int_a^b x(t)^2\,dt\right| \geq 0$$

So $\langle x, x\rangle \geq 0 \quad \forall x \in \underline{X}$

$$\langle x, x\rangle = 0 \longleftrightarrow \int_a^b x(t)\overline{x(t)}\,dt = 0$$

$$\longleftrightarrow \int_a^b |x(t)|^2\,dt = 0$$

$$\longleftrightarrow |x(t)|^2 = 0 \quad \forall\, t \in [a, b]$$

$$\longleftrightarrow |x(t)| = 0 \quad \forall\, t \in [a, b]$$

$$x(t) = 0 \quad \forall\, t \in [a, b]$$

$$\therefore x = 0$$

IP,) $\overline{\langle y, x\rangle} = \overline{\int_a^b y(t)\overline{x(t)}\,dt}$

$$= \int_a^b \overline{y(t)\,\overline{x(t)}}\,dt$$

$$= \int_a^b \overline{\overline{x(t)}}\;\overline{y(t)}\,dt$$

$$= \int_a^b x(t)\,\overline{y(t)}\,dt$$

$$= \langle x, y\rangle$$

$\therefore$ X with $\langle \cdot, \cdot\rangle$ is inner product space

to prove $\overline{X}$ is Hilbert space we take Cauchy sequence in $\overline{X}$ and prove that it's convergent to $\overline{X}$ and limit point belong to $\overline{X}$.

~~Example Let X be the space of inner product~~ vector space of

all continuous functions on

$x(t) = t^{1/2} i$

$y(t) = -t^{1/2} + \cancel{?}$

$z(t) = t i$ on $[0,1]$

find $\langle x+y / z \rangle = \int_a^b (x+y)(t)\, \overline{z(t)}\, dt$

$= \int_0^1 (t^{1/2} i - t^{1/2})\, \overline{t i}\; dt$

$= \int_0^1 (\cancel{?}$

$(-t^{1/2} + t^{1/2} i)\; -ti\; dt$

$= \int_0^1 (t^{3/2} i + t^{3/2})\, dt$

$= \frac{2}{5} t^{5/2} i - \frac{2}{5} t^{5/2} \Big]_0^1$

$= \frac{2}{5} - \frac{2}{5} i$

sol:

Let $\boxed{\alpha = 2+2i}$, $\boxed{x(t) = t + 3t^2 i}$

$z(t) = 1 + t i$

find

$\langle \alpha x, z \rangle = \int_a^b (\alpha x)(t)\, \overline{z(t)}\, dt$

$\alpha x(t) = (2+2i)(t+3t^2 i)$

$= (2t - 6t^2) + (2t + 6t^2) i$