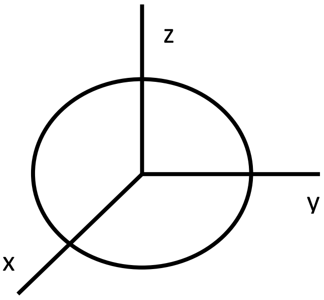
# Triple Integrals in Cylindrical Coordinates

## التكاملات الثلاثية بالإحداثيات الاسطوانية

**Example 2:** By using the **cylindrical coordinate,** find the volume of the spherical (الكرة) with radius 2, i.e., $x^2+y^2+z^2=4$.

**Solution:**

$x^2+y^2+z^2=4 \Rightarrow\ z^2=4-x^2-y^2 \Rightarrow z^2=4-r^2 \Rightarrow z=\pm\sqrt{4-r^2}$




$$\int_0^{2\pi}\int_0^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\ \mathrm{r dr\ d\theta} = \int_0^{2\pi}\int_0^2 z\Big]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}}\ \mathrm{r dr\ d\theta} =$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{4-r^2} - \left[-\sqrt{4-r^2}\right]\ \mathrm{r dr\ d\theta} = \int_0^{2\pi}\int_0^2 2\sqrt{4-r^2}\ \mathrm{r dr\ d\theta}$$

$$-\int_0^{2\pi}\int_0^2 -2\mathrm{r}\sqrt{4-r^2}\ \mathrm{dr\ d\theta} = -\int_0^{2\pi} \left.\frac{[4-r^2]^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 \mathrm{d\theta} = -\frac{2}{3}\int_0^{2\pi} -[4]^{\frac{3}{2}}\ \mathrm{d\theta} = \frac{16}{3}\int_0^{2\pi} \mathrm{d\theta}$$

$$\frac{16}{3}\theta\Big]_0^{2\pi} = \frac{32\pi}{3}\ \ unit^3$$

Second method

$$8\int_0^{\frac{\pi}{2}}\int_0^2\int_0^{\sqrt{4-r^2}} dz\ \ \mathrm{r dr\ d\theta} = 8\int_0^{\frac{\pi}{2}}\int_0^2 z\Big]_0^{\sqrt{4-r^2}}\ \mathrm{r dr\ d\theta} = 8\int_0^{\frac{\pi}{2}}\int_0^2 \sqrt{4-r^2}\ \mathrm{r dr\ d\theta} =$$

$$-4\int_0^{\frac{\pi}{2}} \left.\frac{(4-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 \mathrm{d\theta} = -\frac{8}{3}\int_0^{\frac{\pi}{2}} \left[(0)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]\ \mathrm{d\theta} = -\frac{8}{3}\int_0^{\frac{\pi}{2}} -8\ \mathrm{d\theta} = \frac{64}{3}\theta\Big]_0^{\frac{\pi}{2}} = \frac{32}{3}\pi$$

**Example 3:** By using the **cylindrical coordinate,** find the volume bounded by paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

**Solution:**

$z = x^2 + y^2 \Rrightarrow \; z = r^2$

$$4\int_0^{\frac{\pi}{2}}\int_0^{3}\int_{r^2}^{9} dz \text{ rdr d}\theta = 4\int_0^{\frac{\pi}{2}}\int_0^{3} z]_{r^2}^{9} \text{ rdr d}\theta = 4\int_0^{\frac{\pi}{2}}\int_0^{3} (9-r^2) \text{ rdr d}\theta$$

$$-2\int_0^{\frac{\pi}{2}}\int_0^{3} -2\text{r}\,(9-r^2)\text{ dr d}\theta$$

$$= -2\int_0^{\frac{\pi}{2}} \frac{(9-r^2)^2}{2}\Bigg]_0^3 \text{ d}\theta = -\int_0^{\frac{\pi}{2}} -(9)^{\,2}\;\text{ d}\theta = 81\;\theta]_0^{\frac{\pi}{2}} = \frac{81}{2}\pi$$

Second method

$$\int_0^{2\pi}\int_0^{3}\int_{r^2}^{9} dz \text{ rdr d}\theta = \int_0^{2\pi}\int_0^{3} z]_{r^2}^{9} \text{ rdr d}\theta = \int_0^{2\pi}\int_0^{3} (9-r^2) \text{ rdr d}\theta$$

$$-\frac{1}{2}\int_0^{2\pi}\int_0^{3} -2\text{r}\,(9-r^2)\text{dr d}\theta = -\frac{1}{2}\int_0^{2\pi} \frac{(9-r^2)^2}{2}\Bigg]_0^3 \text{ d}\theta =$$

$$= -\frac{1}{4}\int_0^{2\pi} -(9)^{\,2}\text{ d}\theta = \frac{81}{4}\int_0^{2\pi}\text{ d}\theta = \frac{81}{4}\;\theta]_0^{2\pi} = \frac{81}{2}\pi$$

---

**Example 4:** By using the **cylindrical coordinate,** find the volume of the solid region cut from the spherical (الكرة) $x^2 + y^2 + z^2 = 16$ by cylindrical $x^2 + y^2 = 4$ .

**Solution:**

$x^2 + y^2 + z^2 = 16 \Rrightarrow \; z^2 = 16 - x^2 - y^2 \Rrightarrow z = \pm\sqrt{16 - r^2}$

$$8\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{\sqrt{16-r^2}} dz \text{ rdr d}\theta = 8\int_0^{\frac{\pi}{2}}\int_0^{2} z]_0^{\sqrt{16-r^2}} \text{ rdr d}\theta = 8\int_0^{\frac{\pi}{2}}\int_0^{2} \sqrt{16-r^2}\text{ rdr d}\theta =$$

$$= -4\int_0^{\frac{\pi}{2}} \frac{(16-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^2 \text{ d}\theta = -\frac{8}{3}\int_0^{\frac{\pi}{2}} (12)^{\frac{3}{2}} - (16)^{\frac{3}{2}}\text{ d}\theta =$$

$$= -\frac{8}{3}(12)^{\frac{3}{2}}\theta - (16)^{\frac{3}{2}}\theta]_0^{\frac{\pi}{2}} = -\frac{4}{3}\left[(12)^{\frac{3}{2}} - (16)^{\frac{3}{2}}\right]\pi$$

**Example 5:** Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder (اسطوانة) $x^2 + y^2 = 4$, bounded above by the paraboloid ( الجسم المكافئ الدوراني) $z = x^2 - y^2$ , and bounded below by the $xy$ −plane

**Solution:**

$$\boldsymbol{M} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} \delta\, dz\, r dr\, d\theta = \int_0^{2\pi}\int_0^{2} Z|_0^{r^2}\; r dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r^3\; dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big]_0^2\; d\theta = \int_0^{2\pi} 4\, d\theta = \; 4\theta|_0^{2\pi} = \; 8\pi \quad \textbf{mass}$$

$$\boldsymbol{M_{yz}} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} \boldsymbol{x}\, dz\, r dr\, d\theta = \int_0^{2\pi}\int_0^{2} (r\cos\theta)\;\; Z|_0^{r^2}\; r dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (r\cos\theta)\, r^3\, dr\, d\theta = \int_0^{2\pi}\int_0^{2} \cos\theta\;\; r^4\, dr\, d\theta$$

$$= \int_0^{2\pi} \cos\theta\; \frac{r^5}{5}\Big|_0^2\;\; d\theta = \frac{32}{5}\int_0^{2\pi} \cos\theta\;\, d\theta = \frac{32}{5}\; \sin\theta]_0^{2\pi} = 0, \quad \therefore \quad M_{yz} = 0$$

$$\boldsymbol{M_{xz}} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} \boldsymbol{y}\, dz\, r dr\, d\theta = \int_0^{2\pi}\int_0^{2} (r\sin\theta)\;\; Z|_0^{r^2}\; r dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (r\sin\theta)\, r^3\, dr\, d\theta = \int_0^{2\pi}\int_0^{2} \sin\theta\;\; r^4\, dr\, d\theta$$

$$\int_0^{2\pi} \sin\theta\; \frac{r^5}{5}\Big|_0^2\;\; d\theta = \frac{32}{5}\int_0^{2\pi} \sin\theta\;\, d\theta = -\frac{32}{5}\; \cos\theta|_0^{2\pi} = 0, \quad \therefore \quad M_{xz} = 0$$

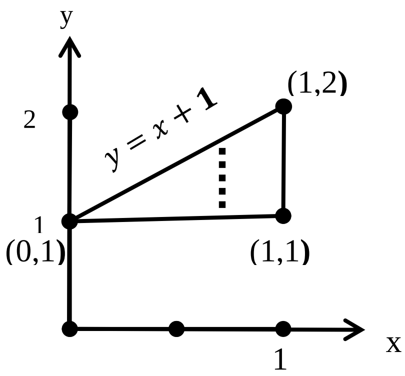
$$\boldsymbol{M_{xy}} = \int_0^{2\pi}\int_0^{2}\int_0^{r^2} \boldsymbol{z}\, dz\, r dr\, d\theta = \int_0^{2\pi}\int_0^{2} \frac{z^2}{2}\Big|_0^{r^2}\;\; r dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{2} r^5\;\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{r^6}{6}\Big|_0^2\;\; d\theta = \frac{1}{12}\int_0^{2\pi} 64\, d\theta = \frac{64}{12}\int_0^{2\pi} d\theta \;\; = \frac{32\pi}{3}, \quad \therefore \quad M_{xy} = \frac{32\pi}{3}$$

***Center of mass is***

$$\bar{X} = \frac{M_{yz}}{M} = \frac{0}{8\pi} = 0\;, \qquad \bar{y} = \frac{M_{xz}}{M} = \frac{0}{8\pi} = 0\,, \qquad \bar{\boldsymbol{Z}} = \frac{M_{xy}}{M} = \frac{\frac{32\pi}{3}}{8\pi} = \frac{4}{3}$$

**Q.1:** Find the mas, first moments, and center of mass of triangular with vertices $(0,1),(1,1),(1,2)$ , and the density at $(x,y)$ is $\delta(x,y)=1$.




**Solution:**

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0} \implies \frac{x-0}{1-0}=\frac{y-1}{2-1} \implies y = x + 1$$

**Mass**

$$M=\iint_R \delta(x,y)\,dA$$

$$M=\int_0^1\int_1^{x+1} dydx=\int_0^1 y|_1^{x+1}\,dx=\int_0^1 (x+1-1)\,dx=\int_0^1 x\,dx=\left.\frac{x^2}{2}\right|_0^1=\frac{1}{2}$$

**First moments**

$$M_x=\int_0^1\int_1^{x+1} ydydx=\int_0^1 \left.\frac{y^2}{2}\right|_1^{x+1} dx=\int_0^1 \left(\frac{(x+1)^2}{2}-\frac{1}{2}\right) dx=\left.\frac{(x+1)^3}{6}-\frac{x}{2}\right|_0^1=$$

$$\frac{8}{6}-\frac{1}{2}-\left[\frac{1}{6}\right]=\frac{7}{6}-\frac{1}{2}=\frac{7-3}{6}=\frac{4}{6}=\frac{2}{3}$$

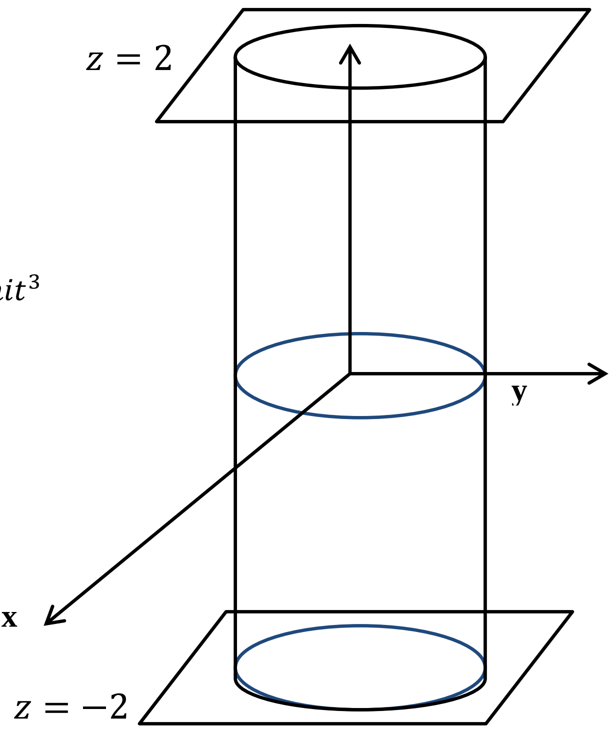
$$M_y=\int_0^1\int_1^{x+1} xdydx=\int_0^1 x\,y|_1^{x+1}\,dx=\int_0^1 x\,(x+1-1)\,dx=\int_0^1 x^2\,dx=\left.\frac{x^3}{3}\right|_0^1=$$

$$=\frac{1}{3}$$

**Center of mass**

$$\bar{X}=\frac{M_Y}{M}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}\,, \qquad \bar{Y}=\frac{M_x}{M}=\frac{\frac{2}{3}}{\frac{1}{2}}=\frac{4}{3}$$

**Q.2:** By using the **cylindrical coordinate,** find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the two planes $z = -2$ $z = 2$ .




**Solution:**

$$V = \int_0^{2\pi}\int_0^{2}\int_{-2}^{2} dz \text{ rdr d}\theta = \int_0^{2\pi}\int_0^{2} z]_{-2}^{2} \text{ rdr d}\theta$$

$$\int_0^{2\pi}\int_0^{2} 4 \text{ rdr d}\theta = = \int_0^{2\pi} 2r^2]_0^2 \text{ d}\theta = \int_0^{2\pi} 8 \text{ d}\theta = 16\pi \ unit^3$$

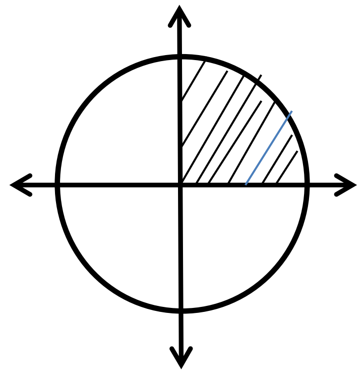
**Second method**

$$V = \mathbf{2}\int_0^{2\pi}\int_0^{2}\int_{0}^{2} dz \text{ rdr d}\theta = \mathbf{2}\int_0^{2\pi}\int_0^{2} z]_0^2 \text{ r dr d}\theta =$$

$$= \mathbf{2}\int_0^{2\pi}\int_0^{2} 2\text{r dr d}\theta = 2\int_0^{2\pi} r^2]_0^2 \text{ d}\theta =$$

$$= 2\int_0^{2\pi} 4 \text{ d}\theta = (2)\ 4\theta]_0^{2\pi} = 16\pi \ unit^3$$

**Third method**



$$V = 2(4)\int_0^{\frac{\pi}{2}}\int_0^{2}\int_{0}^{2} dz \text{ rdr d}\theta = 8\int_0^{\frac{\pi}{2}}\int_0^{2} z]_0^2 \text{ rdr d}\theta =$$

$$= 8\int_0^{\frac{\pi}{2}}\int_0^{2} 2\text{r dr d}\theta = 8\int_0^{\frac{\pi}{2}} r^2]_0^2 \text{ d}\theta =$$

$$8\int_0^{\frac{\pi}{2}} 4 \text{ d}\theta = (8)\ 4\theta]_0^{\frac{\pi}{2}} = 16\pi \ unit^3$$