Sol. $$L[(\sinh at)\, f(t)] = L\left[\frac{e^{at}-e^{-at}}{2} f(t)\right] \qquad \left(\because \sinh at = \frac{e^{at}-e^{-at}}{2}\right)$$

$$= L\left[\frac{1}{2} e^{at} f(t) - \frac{1}{2} e^{-at} f(t)\right]$$

$$= \frac{1}{2} L(e^{at} f(t)) - \frac{1}{2} L(e^{-at} f(t)) \qquad \text{(By linearity)}$$

$$= \frac{1}{2} F(s-a) - \frac{1}{2} F(s-(-a)) \qquad \text{(By first shifting theorem)}$$

$$= \mathbf{\frac{1}{2}\,[F(s-a) - F(s+a)].}$$

Also, $$L(\sin 3t) = \frac{3}{s^2+3^2} = \frac{3}{s^2+9},\ s > 0\,.$$

$$\therefore \quad L\,(\sinh 2t \sin 3t) = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right],\ s > 0+2,\ s > 0+(-2)$$

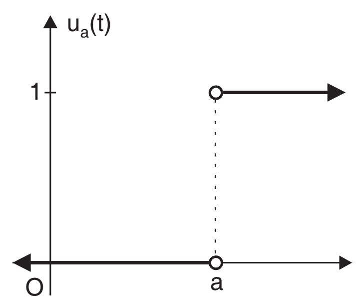
$$= \frac{3}{2}\left[\frac{1}{s^2-4s+13} - \frac{1}{s^2+4s+13}\right],\ s > 2$$

$$= \mathbf{\frac{12s}{s^4+10s^2+169},\ s > 2\,.}$$

1.7. UNIT STEP FUNCTION

Let $a \geq 0$. The function of t taking value 0 if $t < a$ and 1 if $t > a$, is called a **unit step function** and is denoted by $u_a(t)$.

$$\therefore \qquad u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a. \end{cases}$$



1.8. SECOND SHIFTING THEOREM

If $f(t)$ be a function of t for $t \geq 0$ whose Laplace transform $F(s)$ exists then for any constant a (≥ 0), the function $f(t-a)\, u_a(t)$ has the Laplace transform $e^{-as} F(s)$.

Proof. $\quad t < a \;\Rightarrow\; f(t-a)\, u_a(t) = f(t-a)\,.\,0 = 0$

and $\quad t > a \;\Rightarrow\; f(t-a)\, u_a(t) = f(t-a)\,.\,1 = f(t-a)$

$$\therefore \quad f(t-a)\, u_a(t) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

Also, $$\mathrm{F}(s) = \mathrm{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt \qquad ...(1)$$

Now $$\mathrm{L}(f(t-a)\,u_a(t)) = \int_0^\infty e^{-st} f(t-a)\,u_a(t)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\,u_a(t)\,dt + \int_a^\infty e^{-st} f(t-a)\,u_a(t)\,dt$$

$$= \int_0^a e^{-st}\,.\,0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= 0 + \int_0^\infty e^{-s(a+u)} f(u)\,du, \text{where } u = t-a$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)\,du = e^{-sa}\int_0^\infty e^{-st} f(t)\,dt$$

(By replacing u by t)

$$= e^{-as}\,\mathrm{F}(s) \qquad \text{(By using (1))}$$

$\therefore$ **$\mathbf{L(f(t-a)\,u_a(t)) = e^{-as}\,F(s)}$, where $\mathbf{F(s) = L(f(t))}$.**

Remark. The result of above theorem can be easily remembered as follows :

If **$\mathbf{L(f(t)) = F(s)}$, then for any $\mathbf{a \geq 0}$, $\mathbf{L(f(t-a)\,u_a(t)) = e^{-as}\,F(s)}$.**

Example 3. *Find the Laplace transform of the function $t^2\,u_1(t)$.*

Sol. Let $g(t) = t^2\,u_1(t)$.

$\therefore$ $g(t) = ((t-1)+1)^2\,u_1(t) = f(t-1)\,u_1(t)$, where $f(t) = (t+1)^2$.

Now $\mathrm{L}(f(t)) = \mathrm{L}((t+1)^2) = \mathrm{L}(t^2+2t+1)$

$$= \mathrm{L}(t^2) + 2\mathrm{L}(t) + \mathrm{L}(1) = \frac{2!}{s^3} + 2\left(\frac{1!}{s^2}\right) + \frac{1}{s},\ (s>0) = \frac{2+2s+s^2}{s^3},\ s>0.$$

$\therefore$ By **second shifting theorem,**

$$\mathrm{L}(g(t)) = \mathrm{L}(f(t-1))\,u_1(t)) = e^{-1.s}\,\mathrm{L}(f(t))$$

$$= \mathbf{e^{-s}\,.\,\frac{2+2s+s^2}{s^3},\ s>0.}$$

Example 4. *Find the Laplace transform of the following function of t :*

$$g(t) = \begin{cases} 0, & 0<t<\pi/2 \\ \sin t, & t>\pi/2. \end{cases}$$

Sol. Given function $g(t)$ can be written as

$$g(t) = \begin{cases} 0, & 0<t<\pi/2 \\ \cos\left(t-\dfrac{\pi}{2}\right), & t>\dfrac{\pi}{2}. \end{cases}$$

or $$g(t) = \cos\left(t-\frac{\pi}{2}\right)u_{\pi/2}(t)$$

$\therefore$ $$g(t) = f\left(t-\frac{\pi}{2}\right)u_{\pi/2}(t),\ \text{where } f(t) = \cos t$$

Now $\quad L(f(t)) = L(\cos t) = \dfrac{s}{s^2+1^2} = \dfrac{s}{s^2+1}, s > 0$

$\therefore$ By **second shifting theorem,**

$$L(g(t)) = L\left(f\left(t-\frac{\pi}{2}\right)u_{\pi/2}(t)\right)$$

$$= e^{-(\pi/2)s}\, L(f(t)) = \mathbf{\frac{e^{-\pi s/2}\, s}{s^2+1}, s > 0.}$$

Example 5. *Find the Laplace transform of the function* $g(t) = \begin{cases} 0, & 0 < t < 2 \\ t, & t > 2 \end{cases}$ *by using (i) definition (ii) second shifting theorem. Verify that the results are same :*

Sol. We have $g(t) = \begin{cases} 0, & 0 < t < 2 \\ t, & t > 2. \end{cases}$

$$(i)\ L(g) = \int_0^\infty e^{-st}\, g(t)\, dt$$

$$= \int_0^2 e^{-st}\, g(t)\, dt + \int_2^\infty e^{-st}\, g(t)\, dt = \int_0^2 e^{-st}\,.\,0\, dt + \int_2^\infty e^{-st}\, t\, dt$$

$$= 0 + t\,.\frac{e^{-st}}{-s}\Bigg|_2^\infty - \int_2^\infty 1\,.\frac{e^{-st}}{-s}\, dt = -\frac{1}{s}\left[\lim_{T\to\infty}\frac{T}{e^{sT}} - \frac{2}{e^{2s}}\right] + \frac{e^{-st}}{-s^2}\Bigg|_2^\infty$$

$$= -\frac{1}{s}\left[\lim_{T\to\infty}\frac{1}{se^{sT}} - \frac{2}{e^{2s}}\right] - \frac{1}{s^2}\left[\lim_{T\to\infty}\frac{1}{e^{sT}} - \frac{1}{e^{2s}}\right]$$

(Using L'Hospital's Rule)

$$= -\frac{1}{s}\left[0 - \frac{2}{e^{2s}}\right] - \frac{1}{s^2}\left[0 - \frac{1}{e^{2s}}\right] = \frac{2}{se^{2s}} + \frac{1}{s^2e^{2s}} = \mathbf{\frac{2s+1}{s^2\, e^{2s}},\ s > 0.}$$

(ii) $\qquad g(t) = \begin{cases} 0, & 0 < t < 2 \\ (t-2)+2, & t > 2 \end{cases}.$

$\therefore$ $\qquad g(t) = f(t-2)\, u_2(t)$, where $f(t) = t + 2$

Now $\quad L(f(t)) = L(t+2) = L(t) + 2L(1)$

$$= \frac{1!}{s^{1+1}} + 2\left(\frac{1}{s}\right), (s > 0) = \frac{1+2s}{s^2}, s > 0$$

By **second shifting theorem,**

$$L(g(t)) = L(f(t-2)\, u_2(t)) = e^{-2s}\, L(f(t))$$

$$= e^{-2s}\,.\,\frac{1+2s}{s^2}, (s > 0) = \mathbf{\frac{2s+1}{s^2\, e^{2s}}, s > 0.}$$

1.9. CHANGE OF SCALE PROPERTY

Theorem. *If f(t) be a function of t for t ≥ 0 whose Laplace transform F(s) exists, then for any positive constant 'a', the function f(at) has the Laplace transform* $\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$.

Proof. We have $\quad \mathrm{F}(s) = \mathrm{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$.

Now $\quad \mathrm{L}(f(at)) = \int_0^\infty e^{-st} f(at)\, dt = \int_0^\infty e^{-sz/a} f(z) \dfrac{dz}{a}$, where $z = at$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)z} f(z)\, dz = \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t)\, dt$$

(By replacing variable z by t)

$$= \frac{1}{a} \mathrm{F}\left(\frac{s}{a}\right).$$

$\therefore \quad \mathbf{L(f(at)) = \dfrac{1}{a} F\left(\dfrac{s}{a}\right).}$

Remark. The above theorem can be easily remembered as follows :

If $\quad$ L(f(t)) = F(s), then for any a (> 0), $\mathbf{L(f(at)) = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)}$.

Example 6. *Find L(f(λt)) where λ is any positive constant and f(t) is a function of t for $t \geq 0$:*

(*i*) $f(t) = t^n$, $n \in \mathbf{N}$ $\quad$ (*ii*) $f(t) = e^t$ $\quad$ (*iii*) $f(t) = \sinh t$

(*iv*) $f(t) = \cosh t$ $\quad$ (*v*) $f(t) = \sin t$ $\quad$ (*vi*) $f(t) = \cos t$.

Sol. (*i*) $f(t) = t^n$, $n \in \mathbf{N}$

$\therefore \quad \mathrm{L}(f(t)) = \mathrm{L}(t^n) = \dfrac{n!}{s^{n+1}}, s > 0$

By **change of scale property,** $\mathrm{L}(f(\lambda t)) = \dfrac{1}{\lambda} \cdot \dfrac{n!}{(s/\lambda)^{n+1}}, \dfrac{s}{\lambda} > 0$

$$= \frac{\boldsymbol{\lambda^n\, n!}}{\mathbf{s^{n+1}}}, \mathbf{s > 0.} \qquad (\because \ \lambda > 0)$$

(*ii*) $f(t) = e^t$

$$\therefore \quad \mathrm{L}(f(t)) = \mathrm{L}(e^t) = \frac{1}{s-1}, \quad s > 1$$

By **change of scale property,** $\mathrm{L}(f(\lambda t)) = \dfrac{1}{\lambda} \cdot \dfrac{1}{(s/\lambda) - 1}, \quad \dfrac{s}{\lambda} > 1$

$$= \frac{\mathbf{1}}{\mathbf{s} - \boldsymbol{\lambda}}, \mathbf{s} > \boldsymbol{\lambda}.$$

(*iii*) $f(t) = \sinh t$

$\therefore \quad \mathrm{L}(f(t)) = \mathrm{L}(\sinh t) = \dfrac{1}{s^2 - (1)^2}, (s > |1|) = \dfrac{1}{s^2 - 1}, s > 1$

By **change of scale property,** $\mathrm{L}(f(\lambda t)) = \dfrac{1}{\lambda} \cdot \dfrac{1}{(s/\lambda)^2 - 1}, \quad \dfrac{s}{\lambda} > 1$

$$= \frac{\lambda}{\mathbf{s^2} - \lambda^2}, \mathbf{s} > \boldsymbol{\lambda}.$$

(*iv*) $f(t) = \cosh t$

$$\therefore \qquad L(f(t)) = L(\cosh t) = \frac{s}{s^2 - 1}, \; s > 1$$

By **change of scale property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{s/\lambda}{(s/\lambda)^2 - 1}, \quad \frac{s}{\lambda} > 1$

$$= \frac{\mathbf{s}}{\mathbf{s^2} - \lambda^2}, \; \mathbf{s > \lambda.}$$

(*v*) $f(t) = \sin t$

$$\therefore \qquad L(f(t)) = L(\sin t) = \frac{1}{s^2 + 1}, \; s > 0$$

By **change of scalar property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{1}{(s/\lambda)^2 + 1}, \quad \frac{s}{\lambda} > 0$

$$= \frac{\lambda}{\mathbf{s^2} + \lambda^2}, \; \mathbf{s > 0.}$$

(*vi*) $f(t) = \cos t$

$$\therefore \qquad L(f(t)) = L(\cos t) = \frac{s}{s^2 + 1}, \; s > 0$$

By **change of scalar property,** $L(f(\lambda t)) = \frac{1}{\lambda} \cdot \frac{s/\lambda}{(s/\lambda)^2 + 1}, \quad \frac{s}{\lambda} > 0$

$$= \frac{\mathbf{s}}{\mathbf{s^2} + \lambda^2}, \; \mathbf{s > 0.}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L(f(t)) = F(s)$, $s > k$ then for any a, $L(e^{at} f(t)) = F(s - a)$, $s > k + a$.

Rule II. For $a \ge 0$, the unit step function $u(t - a)$ is defined as $u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a. \end{cases}$

Rule III. If $L(f(t)) = F(s)$, then for any $a \ge 0$, $L(f(t - a)\, u(t - a)) = e^{-as} L(f(t))$.

Rule IV. If $L(f(t)) = F(s)$, then for any $a (> 0)$, $L(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right)$.

TEST YOUR KNOWLEDGE

1. Find the Laplace transform of the following functions of t for $t \ge 0$:

(*i*) $e^t t^k$, $k > -1$ (*ii*) $e^{2t} t^4$ (*iii*) $e^t \sinh t$

(*iv*) $e^{3t} \cosh 3t$ (*v*) $e^{2t} \sin t$ (*vi*) $e^{2t} \cos 2t$.

2. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $e^{-4t}\, t^{3/2}$ (*ii*) $e^{-6t}\, t^7$ (*iii*) $e^{-2t} \sinh 6t$

(*iv*) $e^{-5t} \cosh 3t$ (*v*) $e^{-5t} \sin 8t$ (*vi*) $e^{-3t} \cos 4t$.

3. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $e^{3t} \sin^2 t$ (*ii*) $(t + 2)^2\, e^t$

(*iii*) $\sinh 3t \cos^2 t$ (*iv*) $\cosh at \sin at$. (*v*) $e^{-2t} \sin t \cos 3t$.

4. Find the Laplace transform of the following functions of t for $t \geq 0$:

(*i*) $e^{-t} \sin^2 t$ (*ii*) $5e^{2t} \sinh 2t$

(*iii*) $e^{-at} \sinh bt$ (*iv*) $\cosh 4t \sin 6t$.

5. If the Laplace transform of the function $f(t)$ of t for $t \geq 0$ is $F(s)$, then show that

$$L[(\cosh at)\, f(t)] = \frac{1}{2}\,[F(s-a) + F(s+a)].$$

Hence evaluate $L(\cosh 3t \cos 2t)$.

6. Show that $L\{(1 + te^{-t})^3\} = \dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4},\ s > 0.$

7. Find the Laplace transform of the following functions :

(*i*) $(t - 1)\, u_1(t)$ (*ii*) $(5 \cos t)\, u_\pi(t)$.

8. Find the Laplace transform of the function : $g(t) = \begin{cases} 0, & 0 < t < \pi/2 \\ \cos t, & t > \pi/2. \end{cases}$

9. Find the Laplace transform of the function : $g(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & 0 < t < 2\pi/3. \end{cases}$

10. Find the Laplace transform of the function $g(t) = \begin{cases} 0, & 0 < t < 5 \\ t, & t > 5 \end{cases}$ by using (*i*) definition (*ii*) second shifting theorem. Verify that the results are same.

Answers

1. (*i*) $\dfrac{\Gamma(k+1)}{(s-1)^{k+1}},\ s > 1$ (*ii*) $\dfrac{24}{(s-2)^5},\ s > 2$ (*iii*) $\dfrac{1}{s^2 - 2s},\ s > 2$

(*iv*) $\dfrac{s}{s^2 - 6s},\ s > 6$ (*v*) $\dfrac{1}{s^2 - 4s + 5},\ s > 2$ (*vi*) $\dfrac{s-2}{s^2 - 4s + 8},\ s > 2$

2. (*i*) $\dfrac{3\sqrt{\pi}}{4\,(s+4)^{5/2}},\ s > -4$ (*ii*) $\dfrac{5040}{(s+6)^8},\ s > -6$ (*iii*) $\dfrac{6}{s^2 + 4s - 32},\ s > 4$

(*iv*) $\dfrac{s+5}{s^2 + 10s + 16},\ s > -2$ (*v*) $\dfrac{8}{s^2 + 10s + 89},\ s > -5$ (*vi*) $\dfrac{s+3}{s^2 + 6s + 25},\ s > -3$

3. (*i*) $\dfrac{2}{(s-3)(s^2 - 6s + 13)},\ s > 3$ (*ii*) $\dfrac{2}{(s-1)^3}\,[(2s^2 - 2s + 1)],\ s > 1$

(*iii*) $\dfrac{1}{2}\left[\dfrac{s^2 - 6s + 11}{(s-3)(s^2 - 6s + 13)} + \dfrac{s^2 + 6s + 11}{(s+3)(s^2 + 6s + 13)}\right],\ s > 3$ (*iv*) $\dfrac{a(s^2 + 2a^2)}{s^4 + 4a^4},\ s > |a|$

(*v*) $\dfrac{2}{s^2 + 4s + 20} - \dfrac{1}{s^2 + 4s + s},\ s > -2$