Implicit Differentiation

الاشتقاق الضمني

If z is an implicit function of x, y defined by $f(x, y, z) = 0$,Then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad , \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

Example1: Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for the following $3x^2z + y^3 - xyz^3 = 0$.

Solution: first method

The derivative z with respect to x is $\left[3x^2\frac{\partial z}{\partial x} + 6xz\right] - \left[3xyz^2\frac{\partial z}{\partial x} + yz^3\right] = 0$ $3x^2\frac{\partial z}{\partial x} + 6xz - 3xyz^2\frac{\partial z}{\partial x} - yz^3 = 0$ $(3x^2 - 3xyz^2)\frac{\partial z}{\partial x} = yz^3 - 6xz$ $\therefore \frac{\partial z}{\partial x} = \frac{yz^3 - 6xz}{3x^2 - 3xyz^2}$	The derivative z with respect to y is $3x^2\frac{\partial z}{\partial y} + 3y^2 - \left[3xyz^2\frac{\partial z}{\partial y} + xz^3\right] = 0$ $3x^2\frac{\partial z}{\partial y} + 3y^2 - 3xyz^2\frac{\partial z}{\partial y} - xz^3 = 0$ $(3x^2 - 3xyz^2)\frac{\partial z}{\partial y} = xz^3 - 3y^2$ $\therefore \frac{\partial z}{\partial y} = \frac{xz^3 - 3y^2}{3x^2 - 3xyz^2}$

Second method according above rule

$$\frac{\partial f}{\partial x} = 6xz - yz^3$$

$$\frac{\partial f}{\partial y} = 3y^2 - xz^3$$

$$\frac{\partial f}{\partial z} = 3x^2 - 3xyz^2$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{6xz - yz^3}{3x^2 - 3xyz^2}$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{3y^2 - xz^3}{3x^2 - 3xyz^2}$$

Example2: Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for the following $x^2 + 2xz + z^2 - yz - 1 = 0$

Solution: first method

The derivative z with respect to x is	The derivative z with respect to y is
$2x + \left[2x\frac{\partial z}{\partial x} + 2z\right] + 2z\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} = 0$ $2x + 2x\frac{\partial z}{\partial x} + 2z + 2z\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} = 0$ $(2x + 2z - y)\frac{\partial z}{\partial x} = -2x - 2z$ $\therefore \frac{\partial z}{\partial x} = \frac{-2x - 2z}{2x + 2z - y}$	$2x\frac{\partial z}{\partial y} + 2z\frac{\partial z}{\partial y} - \left[y\frac{\partial z}{\partial y} + z\right] = 0$ $2x\frac{\partial z}{\partial y} + 2z\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y} - z = 0$ $(2x + 2z - y)\frac{\partial z}{\partial y} = z$ $\therefore \frac{\partial z}{\partial y} = \frac{z}{2x + 2z - y}$

Second method according above rule

$$\frac{\partial f}{\partial x} = 2x + 2z$$

$$\frac{\partial f}{\partial y} = -z$$

$$\frac{\partial f}{\partial z} = 2x + 2z - y$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{2x+2z}{2x+2z-y} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{-z}{2x+2z-y}$$

Exercise: Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for the following

$x^3 e^{y-z} - y\,sin(x - z) = 0$

$x^3 + y^3 + z^3 + 6xyz = 1$