## ***3rd Lecture***

## Closed Sets and Closure المجموعات المغلقة والانغلاق

**Definition (1.4):** Let $(X, \tau)$ be a topological space and $E \subset X$ we say that $E$ is **closed** iff $d(E) \subset E$.

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**Example (1.9):** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{b, c, d\}, X\}$

Let $E = \{a, b, c\}$, $F = \{b, d\}$ and $H = \{a, d\}$. Determined whether the sets $E$, $F$ and $H$ are closed or not.

**Solution:**

$E = \{a, b, c\}$

$a \notin d(E)$

$b \in d(E)$

$c \in d(E)$

$d \in d(E)$

Now $d(E) = \{b, c, d\} \not\subset E$

$\Rightarrow\ d(E) \not\subset E$

$\Rightarrow\ E$ is not closed.

$F = \{b, d\}$

$c \in d(F)$

$d \in d(F)$

$\Rightarrow\ d(F) = \{c, d\} \not\subset F$

$\Rightarrow\ F$ is not closed

$H = \{a, d\}$

$d(H) = \emptyset \subset H$

$d(H) \subset H \Rightarrow H$ closed.

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**Corollary (1.1):** A subset $E \subset (X, \tau)$ is closed iff $E^c$ is open, i.e.

$$E \text{ closed} \Leftrightarrow E^c \text{ open}$$

**Proof:**

Assume that $E$ is closed. We need to prove that $E^c$ is open.

Let $x \in E^c \Rightarrow x \notin E$, but $E$ is closed

$\Rightarrow \exists\ open\ G_x; x \in G_x \subset E^c$

$\Rightarrow \forall\ x \in E^c, \exists\ open\ G_x; x \in G_x \subset E^c$

$\Rightarrow E^c = \bigcup_{x \in E^c} \{G_x : x \in G_x\}$ open

$\Rightarrow E^c$ is open

Assume that $E^c$ is open. We need to show that $E$ is closed

Let $x \in d(E), x \notin E$

$\Rightarrow x \in d(E), x \in E^c$

But $E^c$ is open and $E \cap E^c = \emptyset$

$\Rightarrow \exists\ open\ G = E^c \ni x; (E \cap E^c) - \{x\} = \emptyset$

$\Rightarrow x \notin d(E) \Rightarrow$ contradiction

$\Rightarrow \forall\ x \in d(E), x \in E \Rightarrow d(E) \subset E \Rightarrow E$ is closed.

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**Corollary (1.2):** If $E$ is closed in $(X, \tau)$, then

$$\exists\ open\ G \ni x \notin E\ such\ that\ x \in G \subset E^c$$

**Proof:**

Assume that the requirement is not true

$\Rightarrow \sim (\exists\ open\ G \ni x, G \subset E^c)$ is true.

$\Rightarrow \forall\ open\ G \ni x; G \not\subset E^c$

$\Rightarrow \forall\ open\ G \ni x; G \cap E \neq \emptyset$

Since $x \notin E$

$\forall\ open\ G; (G \cap E) \setminus \{x\} \neq \emptyset$

$\Rightarrow\ x \in d(E)$

$\Rightarrow\ d(E) \not\subset E$

$\Rightarrow\ E$ is not closed.

$\Rightarrow$ Contradiction

$\Rightarrow$ The requirement is true.

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**Exercise (1.3): (Homework)**

(1) Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}, X\}$. Find

(i) The open sets and the closed sets.

(ii) The sets which are both open and closed.

(iii) The sets which are open but not closed.

(iv) The sets which are closed but not open.

(2) In a topological space if $F$ is closed and $d(F) \subset E \subset F$. Then $E$ is closed.

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