Howo

Evaluate the following:

1. $\int x e^x \, dx$

let $u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = e^x$

$\therefore \int x e^x dx = x e^x - \int e^x du$

$= x e^x - e^x = e^x(x-1) + c$

2. $\int (x+1) e^x \, dx$

let $u = x+1 \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = e^x$

$\therefore \int (x+1) dx = (x+1) e^x - \int e^x dx$

$= (x+1)e^x - e^x$

$= e^x(x+1-1) = x e^x + c$

3. $\int (x^2-2x+5) e^{-x} \, dx$

let $u = x^2-2x+5 \Rightarrow du = 2x-2$

$dv = e^{-x} dx \Rightarrow v = -e^{-x}$

$\therefore \int (x^2-2x+5) e^{-x} dx = -(x^2-2x+5)e^{-x} + \int (2x-2) e^{-x} dx$

$= -(x^2-2x+5)e^{-x} + \underbrace{\int 2x e^{-x} dx}_{I} - \int 2 e^{-x} dx$

let $u_1 = 2x-2 \Rightarrow du = 2dx$

$dv_1 = e^{-x}dx \Rightarrow v = -e^{-x}$

$$\therefore \int (x^2-2x+5)e^{-x}dx = (x^2-2x+5)e^{-x} + 2\int e^{-x}dx - (2x-2)e^{-x} - \int 2e^{-x}dx$$

$$= (x^2-2x+5)e^{-x} + 2e^{-x} - 2xe^{-x} + 2xe^{-x} + 2e^{-x} + c$$

$$= x^2e^{-x} - 2xe^{-x} + 5e^{-x} - 2e^{-x} - 2xe^{-x} + 2xe^{-x} + 2e^{-x} + c$$

$$= e^{-x}[x^2 - 2x + 5 - \cancel{2} - \cancel{2x} + \cancel{2x} + \cancel{2}] + c$$

$$= e^{-x}(x^2 - 2x + 5) + c$$

4. $\int e^{2x}\cos x \, dx$

let $u = e^{2x} \Rightarrow du = 2e^{2x}dx$

$dv = \cos x \, dx \Rightarrow v = \sin x$

$$\therefore \int e^{2x}\cos x \, dx = e^{2x}\sin x - \int 2e^{2x}\sin x \, dx$$

let $u = e^{2x} \Rightarrow du = 2e^{2x}dx$

$dv = \sin x dx \Rightarrow v = -\cos x$

$$\therefore \int e^{2x}\cos x \, dx = e^{2x}\sin x + 2\left[e^{2x}\cos x + \int 2e^{2x}\cos x \, dx\right]$$

let $u = \cos x \Rightarrow du = -\sin x \, dx$

$dv = e^{2x} dx \Rightarrow v = 2e^{2x}$

$$\therefore \int e^{2x} \cos x \, dx = e^{-2x} \sin x - 2\left[-e^{2x} \cos x + \left[2e^{2x} \cos x + \int 2e^{x} \sin x \, dx\right]\right.$$

Since the function take the same Pattern for three stage, we stop and call the Function aperiodic function.

3) Integration of Fractions

let f and g be two function of defined by:

$$R(X) = \frac{f(X)}{g(X)} \quad ; \quad g(X) \neq 0$$

Such that f(x) and g(x) are multiple polynomial, then two cases:-

A) Case one

when the order of the numerator is less than the order of the denominator; that is, the function is proper.

B) Case two

when the order of the numerator is greater or equal than the order of the denominator; that is the function is improper.

For examples of types of functions:-

1. $f(X) = \frac{X+1}{X^3 - 3X}$ proper

2. $f(X) = \frac{X^3 + 5X^2}{X^2 - 2}$ improper

3. $f(X) = \frac{X+1}{X-2}$ improper

4. $f(X) = \frac{X}{X^2 + 2}$ proper

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Improper function :-

If the order of the numerator is greater than or equal to the order of the denominator, that is the function is improper :-

For example :-

1. $\int \frac{x+1}{x-2}\,dx$

To find the integral we use long division as follows :-

$$\begin{array}{r|l} & 1 \\ \hline x-2 & x+1 \\ & \mp x \pm 2 \\ \hline & 3 \end{array}$$

$$\therefore \int \frac{x+1}{x-2}\,dx = \int 1\,dx + \int \frac{3}{x-2}\,dx$$

$$= x + 3\ln|x-2| + c$$

2. $\int \frac{x^3+5x^2}{x^2+2}\,dx$

$$\begin{array}{r|l} & x+5 \\ \hline x^2+2 & x^3+5x^2 \\ & \mp x^3 \quad \mp 2x \\ \hline & 5x^2-2x \\ & \mp 5x^2 \mp 10 \\ \hline & -2x-10 \end{array}$$

$$\therefore \int \frac{x^3+5x^2}{x^2+2}\,dx = \int (x+5)\,dx + \int \left(\frac{-2x-10}{x^2+2}\right)dx$$

$$= \frac{x^2}{2} + 5x - \int \frac{2x}{x^2+2}\,dx - \int \frac{10}{x^2+2}\,dx$$

$$= \frac{x^2}{2} + 5x - \ln|x^2+2| - \frac{10}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$$

3. $\int \frac{x+1}{x+3}\,dx$

$$\begin{array}{r|l} & 1 \\ \hline x+3 & x+1 \\ & \mp x \mp 3 \\ \hline & -2 \end{array}$$

$\therefore \int \frac{x+1}{x+3}\,dx = \int 1\,dx + \int \frac{-2}{x+3}\,dx$

$= x - 2\ln|x+3| + c$

4. $\int \ln(x^2+2)\,dx$

<u>Solu</u> let $u = \ln(x^2+2) \Rightarrow du = \frac{1}{x^2+2} \cdot 2x\,dx$

$dv = dx \Rightarrow v = x$

$\int \ln(x^2+2)\,dx = x\ln(x^2+2) - \int \frac{2x^2}{x^2+2}\,dx$

$$\begin{array}{r|l} & 1 \\ \hline x^2+2 & x^2 \\ & \mp x^2 \mp 2 \\ \hline & -2 \end{array}$$

$\therefore \int \ln(x^2+2)\,dx = x\ln(x^2+2) - 2\left[\int dx - \int \frac{2}{x^2+2}\,dx\right.$

$= x\ln(x^2+2) - 2x + \frac{2}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$

Case two :- Proper function

If the order of the numerator is less than the order of the denominator, then the function is proper , there are several cases:-

A) If the denominator decomposes into real and different roots then:-

1. Factoring the denominator into prime factors:-

$(X+a), (X+b), \dots, (X+d)$

the integration becomes to:

$$= \int \frac{A}{X+a}\,dX + \int \frac{B}{X+b}\,dX \dots +$$

2. Mutiply both sides of expression by the denominator of the fraction to be divided to get rid of fractions:-

3. Solve equation
4. perform integration

Ex: Find the following integral

1. $\int \frac{1}{X^2-4}\,dX$

$$\int \frac{1}{X^2-4}\,dX = \int \frac{A}{X+2}\,dX + \int \frac{B}{X-2}\,dX$$

$$\left[\frac{1}{X^2-4} = \frac{A}{X+2} + \frac{B}{X-2} \right] (X^2-4) \text{ or } (X-2)(X+2)$$

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$$1 = A(X-2) + B(X+2)$$
$$1 = AX - 2A + BX + 2B$$

$$\therefore 1 = 2B - 2A \quad (1)$$

$$0 = AX + BX \quad (2) \quad \div X \Rightarrow A = -B$$

Substituting the value of A into equation 1 we get :-

$$1 = 2B - 2(-B) \Rightarrow B = \frac{1}{4} \text{ and } A = -\frac{1}{4}$$

By substituting the values A and B in the main equation, we get the result of the integration as follows :-

$$\int \frac{1}{X^2-4} dX = \int \frac{-\frac{1}{4}}{X-2} dx + \int \frac{\frac{1}{4}}{X+2} dX$$

$$= -\frac{1}{4} \ln|X-2| + \frac{1}{4} \ln|X+2| + c$$

2. $\displaystyle\int \frac{dX}{X^2+5X+6}$

$$X^2+5X+6 = (X+3)(X+2)$$

$$\therefore \int \frac{1}{X^2+5X+6} dX = \int \frac{A}{X+3} dx + \int \frac{B}{X+2} dX$$

$$\left(\frac{1}{X^2+5X+6} = \frac{A}{X+3} + \frac{B}{X+2} \right) (X^2+5X+6) \text{ or } (X+3)(X+2)$$

$$1 = A(X+2) + B(X+3)$$
$$1 = AX + 2A + BX + 3B$$

$1 = 2A + 3B$ (1)

$0 = XA + XB$ (2) $\div X \Rightarrow A = -B$

$1 = 2A + 3B$

$0 = A + B$ $*2$

$1 = 2A + 3B$

$0 = -2A \mp 2B$

$1 = B \Rightarrow B = 1 \Rightarrow A = -1$

$$\therefore \int \frac{dX}{X^2+5X+6} = \int \frac{-1}{X+3} dx + \int \frac{1}{X+2} dX$$

$$= -\ln|X+3| + \ln|x+2| + C$$

3. $\int \frac{5X-3}{X^2-2X-3} dX$

$X^2 - 2X - 3 = (X-3)(X+1)$

$$\therefore \int \frac{5X-3}{X^2-2X-3} dx = \int \frac{A}{X-3} dx + \int \frac{B}{X+1} dx$$

$$\therefore \left(\frac{5X-3}{X^2-2X-3} = \frac{A}{X-3} + \frac{B}{X+1} \right) * (X^2-2X-3) \text{ or } (X-3)(X+1)$$

$5X - 3 = A(X+1) + B(X-3)$

$5X - 3 = AX + A + BX - 3B$

$5X = AX + BX$ (1) $\div X \Rightarrow A + B = 5$

$-3 = A - 3B$ (2)

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$\because$ $A + B = 5 \quad *3$

$A - 3B = -3$

$3A + 3B = 15$

$A - 3B = -3$

$4A = 12 \Rightarrow A = 3 \Rightarrow B = 2$

$$\because \int \frac{5X - 3}{X^2 - 2X - 3} dx = \int \frac{3}{X-3} dx + \int \frac{2}{X+1} dx$$

$$= 3\ln|X-3| + 2\ln|X+1| + c$$

H.W.

1. $\int \frac{X + 1}{X^2 - 3X + 2} dx \Rightarrow (X-2)(X-1)$

2. $\int \frac{X + 4}{X^3 + 6X^2 + 11X + 6} dx$

$$X^3 + 6X^2 + 11X + 6 = (X+1)(X^2 + 5X + 6)$$

$$(X+1)(X+3)(X+2)$$