1.3 Banach space

Definiation complete space.

A normed space $(X, \|\cdot\|)$ is called complete space if every Cauchy sequence in X is convergent to an element in X.

Definition Banach space

A complete normed space is called Banach space

EX R is a Banach space
see [Theorem 6.3, P. 37] in كتاب مقدمة في التحليل الرياضي

EX show that $(\mathbb{C}, |\cdot|)$ is Banach space over the field R

$\|x\| = \|x_1 + x_2 i\| = |x_1| + |x_2|$

Solution let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in R, such that $x_n = a_n + b_n i$ where $a_n, b_n \in R$

$(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ is a two sequence in R

$\because$ $(X_n)_{n=1}^{\infty}$ is Cauchy sequence, for every $\epsilon > 0$ there exists $k_1 \in N$ such that.

$\| x_n - x_m \| < \epsilon \quad \forall n, m > k_1$

~~$\|x_n - x_m\|$~~ $x_n = a_n + b_n i$

$x_m = a_m + b_m i$

$\| x_n - x_m \| = \|(a_n + b_n i) - (a_m + b_m i)\|$

$= \| (a_n - a_m) + (b_n - b_m) i \|$

$= |a_n - a_m| + |b_n - b_m| < \epsilon \quad \forall n, m > k_1$

we get that $|a_n - a_m| < \epsilon \ \forall n, m > k_1$

$|b_n - b_m| < \epsilon \ \forall n, m > k_1$

so $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence in R

$\because$ R is Banach space each Cauchy sequence in R is convergent let $a_n \longrightarrow a_0$

$b_n \longrightarrow b_0$

that is mean for $\frac{\epsilon}{2} > 0$, $\exists$ $k_2 \in N$ such that

$|a_n - a_0| < \frac{\epsilon}{2} \quad \forall n > k_2$

and for $\frac{\epsilon}{2} > 0$, $\exists$ $k_3 \in N$ such that

$|b_n - b_0| < \frac{\epsilon}{2} \quad \forall n > k_3$

Let $K = \max\{k_1, k_2\}$, $x_0 = a_0 + b_0 i$
it is clearly that $x_0 \in \mathbb{C}$
is the convergent point which $(x_n)_{n\geq 1}^{\infty}$
will be convergent to it.

now we to show that $x_n \longrightarrow x_0$

$\|x_n - x_0\| = \|(a_n + b_n i) - (a_0 + b_0 i)\|$

$= \|(a_n - a_0) + (b_n - b_0) i\|$

$= |a_n - a_0| + |b_n - b_0|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n > K$

$\therefore \|x_n - x_0\| < \epsilon \quad \forall n > K$

so $(x_n)_{n\geq 1}^{\infty}$ is convergent to x_0

$\therefore$ C over R is a Banach space.

Ex The space $(C[a,b], \|\cdot\|_{\infty})$ is Banach space with

$\|f\|_{\infty} = \sup_{x \in [a,b]}\{|f(x)|\} \qquad f \in C[a,b]$

proof: Let $(f_n)_{n\geq 1}^{\infty}$ be a Cauchy sequence in $C[a,b]$, Then for every $\epsilon > 0$, there exist $k_1 \in N$ such that $\| f_n - f_m \|_\infty < \epsilon \quad \forall n, m > k_1$

$$\| f_n - f_m \|_\infty = \sup_{x \in [a,b]} | f_n(x) - f_m(x) | < \epsilon$$

~~This is mean~~ This is mean

$$| f_n(x) - f_m(x) | < \epsilon \quad \forall m, n > k_1$$

~~Since~~ So $(f_n(x))_{n=1}^\infty$ is a Cauchy sequence in R

$\because R$ is Banach, so $(f_n(x))_{n=1}^\infty$ is a convergent sequence in R, let it is convergent to $f_0(x)$

$$f_n(x) \longrightarrow f_0(x) \quad \forall x \in [a, b]$$

for any $\epsilon > 0$, there exists $k \in N$ such that

$$| f_n(x) - f_0(x) | < \epsilon \quad \forall n > k_2$$

now, we ~~consider~~ to prove that $f_n \to f_0$

$$\| f_n - f_0 \|_\infty = \sup_{x \in [a,b]} | f_n(x) - f_0(x) |$$

$$< \sup_{x \in [a,b]} \epsilon = \epsilon \quad \forall n > k_2$$

$\therefore \| f_n - f_0 \|_\infty < \epsilon \quad \forall n > k_2$

So $f_n \longrightarrow f_0$

where $f_0 : [a, b] \longrightarrow R$

f_0 is a function has domain $[a,b]$ and ~~...~~ codomain in $\mathbb{R}$, we must show that f_0 is a continuous, therefore

Let $(x_n)_{n=1}^{\infty}$ be ~~a~~ a convergent sequence to x_0 in $[a,b]$, $x_n \to x_0$ and $x_0 \in [a,b]$

if $f_0(x_n) \to f_0(x_0)$ then f_0 is continuous function

$$|f_0(x_n) - f_0(x_0)| = |f_0(x_n) - f_n(x_n) + f_n(x_n) - f_n(x_0) + f_n(x_0) - f_0(x_0)|$$

$$\leq |f_0(x_n) - f_n(x_n)| + |f_n(x_n) - f_n(x_0)| + |f_n(x_0) - f_0(x_0)|$$

$$= \underbrace{|f_n(x_n) - f_0(x_n)|}_{1} + \underbrace{|f_n(x_n) - f_n(x_0)|}_{2} + \underbrace{|f_n(x_0) - f_0(x_0)|}_{3}$$

for 1 $\because$ $f_n(x) \to f_0(x)$ $\forall x \in [a,b]$

so $f_n(x_n) \to f_0(x_n)$ $\forall x_n \in [a,b]$

for $\frac{\epsilon}{3} > 0$, $\exists k_3 \in \mathbb{N}$ such that

$|f_n(x_n) - f_0(x_n)| < \frac{\epsilon}{3}$ $\forall n > k_3$

for 2 $f_n \in C[a,b]$ which is a continuous function

$\because x_n \to x_0$ which is implies $f_n(x_n) \to f_n(x_0)$

that is mean, for $\frac{\epsilon}{3} > 0$ there exist $k_4 \in N$ such that

$$|f_n(x_n) - f_n(x_0)| < \frac{\epsilon}{3} \quad \forall n > k_4$$

for ③ $\because f_n(x) \to f_0(x) \quad \forall x \in [a,b]$

$\therefore f_n(x_0) \to f_0(x_0)$ for $x_0 [a,b]$

Then for $\frac{\epsilon}{3} > 0$, there exist $k_5 \in N$ such that

$$|f_n(x_0) - f_0(x_0)| < \frac{\epsilon}{3} \quad \forall n > k_5$$

Let $k = \max\{k_3, k_4, k_5\}$, so

$$|f_0(x_n) - f_0(x_0)| \leq |f_n(x_n) - f_0(x_n)| + |f_n(x_n) - f_n(x_0)| + |f_n(x_0) - f_0(x_0)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \forall n > k$$

$\therefore f_0$ is continuous function

so $f_0 \in C[a,b]$

That is mean $C[a,b]$ is a Banach space

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Open set, Closed set, Neighborhood

There is a considerable of concepts which is play a role in connection with (metric space, normed space) and Topological space, we start with concept open set.

Definition Open set.

Let $x_0 \in X$ (with X is normed (metric space) and real number $r > 0$. we define

① open ball

نصف قطر   المركز

$$B_r(x_0) = \{ x \in X \text{ , } d(x, x_0) < r \text{ or } \|x - x_0\| < r \}$$

in metric

in normed space

② closed ~~set~~ ball

$$\overline{B}_r(x_0) = \{ x \in X \text{ , } d(x, x_0) \le r \text{ or } \|x - x_0\| \le r \}$$

in metric

in normed

③ sphere

$$S_r(x_0) = \{ x \in X \text{ , } d(x, x_0) = r \text{ or } \|x - x_0\| = r \}$$

x_0 is called the center ~~of the~~

r is called the radius

we see $S_r(x_0) = \overline{B}_r(x_0) - B_r(x_0)$