

Advanced Calculus (2)

Triple Integrals in Spherical Coordinates

التكاملات الثلاثية بالأحداثيات الكروية

Example 1: Convert the point $(8, -\frac{\pi}{3}, -\frac{\pi}{6})$ from spherical to rectangular coordinates.

Solution: $(\rho, \phi, \theta) \Rightarrow (x, y, z)$

$$x = \rho \sin\phi \cos\theta \Rightarrow x = 8 \sin(-\frac{\pi}{3}) \cos(-\frac{\pi}{6})$$

$$\Rightarrow x = -8 \sin(\frac{\pi}{3}) \cos(\frac{\pi}{6}) \Rightarrow x = -8 \frac{\sqrt{3}}{2} \frac{1}{2} \Rightarrow x = -6$$

$$y = \rho \sin\phi \sin\theta \Rightarrow y = 8 \sin(-\frac{\pi}{3}) \sin(-\frac{\pi}{6})$$

$$\Rightarrow y = +8 \sin(\frac{\pi}{3}) \sin(\frac{\pi}{6}) \Rightarrow y = 8 \frac{\sqrt{3}}{2} \frac{1}{2} \Rightarrow y = 2\sqrt{3}$$

$$z = \rho \cos\phi \Rightarrow z = 8 \cos(-\frac{\pi}{3}) \Rightarrow z = 8 (\frac{1}{2}) \Rightarrow z = 4$$

$$\therefore (8, -\frac{\pi}{3}, -\frac{\pi}{6}) \Rightarrow (-6, 2\sqrt{3}, 4)$$

Example 2: By using the **spherical coordinate**, find the volume of the spherical with center origin point and radius equal to 6

Solution:

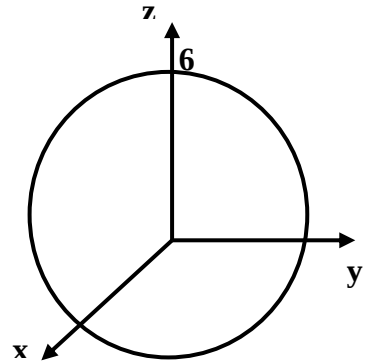
$$V = 8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^6 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^3}{3} \right]_0^6 \sin\phi \, d\phi \, d\theta =$$

$$8 \frac{(6)^3}{3} \int_0^{\frac{\pi}{2}} [-\cos\phi]_0^{\frac{\pi}{2}} d\theta = 8 \frac{(6)^3}{3} \int_0^{\frac{\pi}{2}} d\theta = 8 \frac{(6)^3}{3} \frac{\pi}{2} = 4 \frac{(6)^3 \pi}{3}$$

Based on cylindrical coordinates

$$8 \int_0^{\frac{\pi}{2}} \int_0^6 \int_0^{\sqrt{36-r^2}} dz \, r \, dr \, d\theta = 8 \int_0^{\frac{\pi}{2}} \int_0^6 z \Big|_0^{\sqrt{36-r^2}} r \, dr \, d\theta = 8 \int_0^{\frac{\pi}{2}} \int_0^6 \sqrt{36-r^2} \, r \, dr \, d\theta =$$



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$$-4 \int_0^{\frac{\pi}{2}} \left[\frac{(36 - r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^6 d\theta = -\frac{8}{3} \int_0^{\frac{\pi}{2}} \left[(0)^{\frac{3}{2}} - (36)^{\frac{3}{2}} \right] d\theta = \frac{8(6)^3}{3} \int_0^{\frac{\pi}{2}} d\theta = 4 \frac{(6)^3 \pi}{3}$$

Example 3: By using the **spherical coordinate**, find the solid volume above of the $z = \sqrt{x^2 + y^2}$, below the sphere $x^2 + y^2 + z^2 = 16$

Solution: $z = \sqrt{x^2 + y^2}$

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta}$$

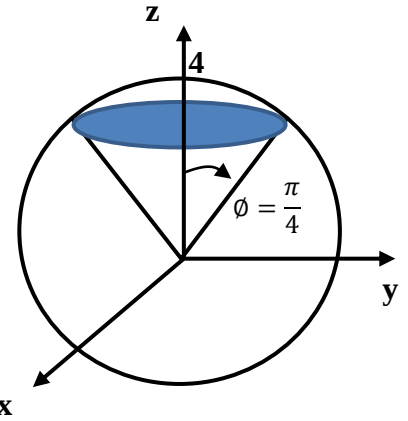
$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi [\cos^2 \theta + \sin^2 \theta]}$$

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi}$$

$$\rho \cos \phi = \rho \sin \phi \Rightarrow \frac{\sin \phi}{\cos \phi} = 1 \Rightarrow \tan \phi = 1$$

$$\therefore \phi = \frac{\pi}{4}$$

$$\begin{aligned} V &= 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3} \right]_0^4 \sin \phi \, d\phi \, d\theta = \\ &= 4 \frac{(4)^3}{3} \int_0^{\frac{\pi}{2}} [-\cos \phi]_0^{\frac{\pi}{4}} d\theta = -4 \frac{(4)^3}{3} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{2}} - 1 \right) d\theta = \end{aligned}$$



Example 4: Find the mass of the sphere with center origin points and radius equal 2, and the density at (x, y) is $\delta(x, y) = 1$.

Solution: $x^2 + y^2 + z^2 = 4$

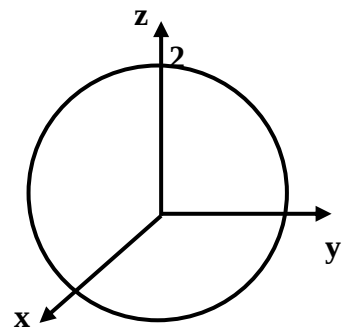
1. Based on rectangular coordinates.

$$M = 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \delta \, dz \, dy \, dx =$$

2. Based on cylindrical coordinates.

$$M = 8 \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{\sqrt{4-r^2}} \delta \, dz \, r \, dr \, d\theta =$$

3. Based on spherical coordinates



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$$M = 8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 \delta \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta =$$

Example 5: By using the **spherical coordinate**, find the solid volume above of the cone (مخروط) $z^2 = x^2 + y^2$, below the sphere $x^2 + y^2 + z^2 = 9$

Solution:

$$\rho^2 = x^2 + y^2 + z^2 = 9 \Rightarrow \rho = 3$$

The intersection of cone and sphere

$$x^2 + y^2 + z^2 = 9 \dots (1)$$

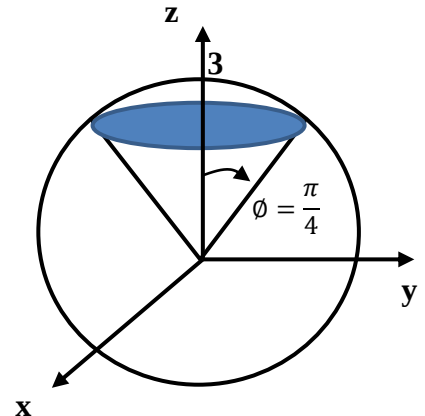
$$x^2 + y^2 = z^2 \dots (2)$$

Substitute Eq.(2) in Eq. (1) we get

$$z^2 + z^2 = 9 \Rightarrow 2z^2 = 9 \Rightarrow z = \frac{3}{\sqrt{2}}$$

Since

$$z = \rho \cos \phi \Rightarrow \frac{3}{\sqrt{2}} = 3 \cos \phi \Rightarrow \cos \phi = \frac{1}{\sqrt{2}} \\ \therefore \phi = \frac{\pi}{4}$$



Or

$$z^2 = x^2 + y^2$$

$$\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta$$

$$\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi [\cos^2 \theta + \sin^2 \theta]$$

$$\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi$$

$$\cos \phi = \sin \phi$$

$$\frac{\sin \phi}{\cos \phi} = 1 \Rightarrow \tan \phi = 1$$

$$\therefore \phi = \frac{\pi}{4}$$

$$V = 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3} \right]_0^3 \sin \phi \, d\phi \, d\theta = \\ = 4 \frac{(4)^3}{3} \int_0^{\frac{\pi}{2}} [-\cos \phi]_0^{\frac{\pi}{4}} d\theta = -4 \frac{(4)^3}{3} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{2}} - 1 \right) d\theta =$$