4. (*i*) $\dfrac{2}{(s+1)(s^2+2s+5)}$, $s > -1$ (*ii*) $\dfrac{10}{s^2-4s}$, $s > 4$

(*iii*) $\dfrac{b}{s^2+2as+a^2-b^2}$, $s > |b| - a$ (*iv*) $\dfrac{6(s^2+52)}{s^4+40s^2+2704}$, $s > 4$

5. $\dfrac{s^3-5s}{s^4-10s^2+169}$, $s > 3$ **7.** (*i*) $\dfrac{e^{-s}}{s^2}$, $s > 0$ (*ii*) $\dfrac{-5se^{-\pi s}}{s^2+1}$, $s > 0$

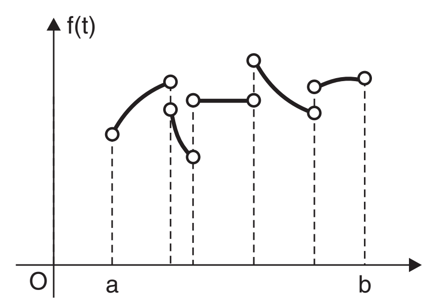
8. $-\dfrac{e^{-\pi s/2}}{s^2+1}$, $s > 0$ **9.** $\dfrac{e^{-2\pi s/3}s}{s^2+1}$, $s > 0$ **10.** $\dfrac{(5s+1)e^{-5s}}{s^2}$, $s > 0$

Hint

8. Use $\cos t = -\sin\left(t - \dfrac{\pi}{2}\right)$.

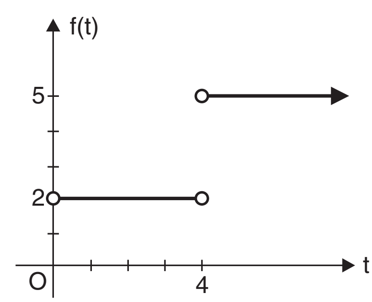
1.10. PIECEWISE CONTINUOUS FUNCTION

A function $f(t)$ is called **piecewise continuous (**or **sectionally continuous)** on a finite interval $[a, b]$ if this interval can be divided into a finite number of sub-intervals such that (*i*) $f(t)$ is continuous in the interior of each of these sub-intervals and (*ii*) $f(t)$ approaches a finite limit as t approaches either endpoint of each of the sub-interval from its interior.



Thus a piecewise continuous function has finitely many jumps as discontinuities. Clearly every continuous function is a piecewise continuous function. Also, if $f(t)$ is a piecewise continuous function on $[a, b]$, then $f(x)$ is integrable on $[a, b]$.

For example, let $f(t) = \begin{cases} 2, & 0 < t < 4 \\ 5, & t > 4. \end{cases}$



The function $f(x)$ is piecewise continuous on every finite interval $[0, b]$, where b is a positive real number.

1.11. EXISTENCE THEOREM FOR LAPLACE TRANSFORMS

The Laplace transform of any arbitrary function may not exist. For the existence of the Laplace transform of a function, the function is expected to fulfil certain conditions. In the ensuing theorem, we shall establish the criterion for the existence of Laplace transform of a function.

Theorem. *Let f(t) be a real function such that*

(*i*) *f(t) is piece wise continuous on every finite interval in the range* $t \geq 0$ *and*

(*ii*) *there exists constants k and M such that* $|f(t)| \leq Me^{kt}$ *for* $t \geq 0$, *then prove that the Laplace transform of f(t) exists for all* $s > k$.

Proof. Since $f(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$, $f(t)$ is piece wise continuous on the finite interval [0, T] for T > 0.

Also e^{-st} is continuous on the finite interval [0, T] for T > 0

$\therefore$ $e^{-st} f(t)$ is piecewise continuous on [0, T] for T > 0.

$\therefore$ $e^{-st} f(t)$ is integrable on [0, T] for T > 0.

$$\therefore \quad \int_0^T e^{-st} f(t)\,dt \text{ exists for all T > 0.}$$

Now,
$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt$$

and
$$\left| \int_0^\infty e^{-st} f(t)\,dt \right| \leq \int_0^\infty \left| e^{-st} f(t) \right| dt = \int_0^\infty e^{-st} |f(t)|\,dt$$

$$\leq \int_0^\infty e^{-st} \,.\, \mathrm{M}e^{kt}\,dt = \mathrm{M}\int_0^\infty e^{(k-s)t}\,dt = \mathrm{M}\lim_{T\to\infty}\left[\frac{e^{(k-s)t}}{k-s}\right]_0^T$$

$$= \frac{\mathrm{M}}{k-s}\lim_{T\to\infty}\left[\frac{1}{e^{(s-k)T}} - 1\right] = \frac{\mathrm{M}}{k-s}[0-1] \quad \text{for } s > k$$

$$= \frac{\mathrm{M}}{s-k} \quad \text{for } s > k\,.$$

$$\therefore \quad \int_0^\infty e^{-st} f(t)\,dt \leq \frac{\mathrm{M}}{s-k} \quad \text{for } s > k$$

$$\therefore \quad \mathrm{L}(f) = \int_0^\infty e^{-st} f(t)\,dt \text{ exists for } s > k.$$

1.12. LAPLACE TRANSFORMS OF DERIVATIVES

Theorem 1. *(Laplace transform of the first derivative). Let $f(t)$ be a real function such that*

(i) $f(t)$ is continuous for all $t \geq 0$

(ii) there exists constants k and M such that $|f(t)| \leq Me^{kt}$ for $t \geq 0$

(iii) $f'(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$,

then the Laplace transform of the derivative $f'(t)$ exists and

$$L(f') = sL(f) - f(0) \quad \text{for } s > k.$$

Proof. By definition

$$\mathrm{L}(f') = \lim_{T\to\infty}\int_0^T e^{-st} f'(t)\,dt, \text{ provided this limit has some finite value.}$$

Let T be any positive number.

Using (iii), $f'(t)$ is piecewise continuous on [0, T].

$\therefore$ $f'(t)$ has at most a finite number of discontinuities, say $t_1, t_2, \ldots\ldots, t_n$, where

$$0 < t_1 < t_2 < \ldots\ldots < t_n \leq T$$

$$\therefore \quad \int_0^T e^{-st} f'(t)\,dt = \int_0^{t_1} e^{-st} f'(t)\,dt + \ldots\ldots + \int_{t_n}^T e^{-st} f'(t)\,dt$$

The integrand of each of the integrals on the right is continuous.

$\therefore$ By using integration by parts, we have

$$\int_0^T e^{-st} f'(t)\, dt = \left[e^{-st} f(t)\right]_0^{t_1} + s\int_0^{t_1} e^{-st} f(t)\, dt + \ldots\ldots + \left[e^{-st} f(t)\right]_{t_n}^{T} + s\int_{t_n}^{T} e^{-st} f(t)\, dt$$

$$= \left[e^{-st_1} f(t_1) - f(0)\right] + s\int_0^{t_1} e^{-st} f(t)\, dt + \ldots\ldots$$

$$+ \left[e^{-sT} f(T) - e^{-st_n} f(t_n)\right] + s\int_{t_n}^{T} e^{-st} f(t)\, dt$$

$$= e^{-sT} f(T) - f(0) + s\left[\int_0^{t_1} e^{-st} f(t)\, dt + \ldots\ldots + \int_{t_n}^{T} e^{-st} f(t)\, dt\right]$$

$\therefore$
$$\int_0^T e^{-st} f'(t)\, dt = e^{-sT} f(T) - f(0) + s\int_0^T e^{-st} f(t)\, dt$$

($\because$ f is continuous for $t \geq 0$)

$\therefore$
$$\lim_{T\to\infty}\int_0^T e^{-st} f'(t)\, dt = \lim_{T\to\infty} e^{-sT} f(T) - f(0) + \lim_{T\to\infty} s\int_0^T e^{-st} f(t)\, dt \quad \ldots(1)$$

Now using (*ii*), $e^{-sT} f(T) \leq e^{-sT}\,|f(T)| \leq e^{-sT}\,.\,Me^{kT} = \dfrac{M}{e^{(s-k)T}}$

$\therefore$ $\displaystyle\lim_{T\to\infty} e^{-sT} f(T) = 0$ for $s > k$

Also $\displaystyle\lim_{T\to\infty} s\int_0^T e^{-st} f(t)\, dt = sL(f)$

$\therefore$ (1) $\Rightarrow$ $\displaystyle\lim_{T\to\infty}\int_0^T e^{-st} f'(t)\, dt = 0 - f(0) + sL(f)$.

$\therefore$ **L(f′) = sL(f) – f(0) for s > k**.

ILLUSTRATIVE EXAMPLES

Example 1. *Using* $L(t^n) = \dfrac{n!}{s^{n+1}}$ *, find the value of* $L(t^{n+1})$.

Sol. Let $f(t) = t^{n+1}$.

$\therefore$ $f'(t) = (n + 1)\, t^n$.

We have $L(f') = sL(f) - f(0)$.

$\therefore$ $L((n + 1)\, t^n) = sL(t^{n+1}) - (0)^{n+1} \Rightarrow (n + 1)\, L(t^n) = sL(t^{n+1})$

$\therefore$
$$L(t^{n+1}) = \frac{n+1}{s} L(t^n) = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{\mathbf{(n+1)!}}{\mathbf{s^{n+2}}}.$$

Example 2. *Find the Laplace transform of the function* $\sin^2 at \cos at,\ t \geq 0$.

Sol. Let $f(t) = \sin^3 at$.

$\therefore$ $f'(t) = 3\sin^2 at\ .\ a\cos at = 3a\cos at\sin^2 at$

We have $\quad L(f') = sL(f) - f(0).$

$\therefore \quad L(3a \cos at \sin^2 at) = sL(\sin^3 at) - \sin^3 0$

$\Rightarrow \quad 3a\, L(\cos at \sin^2 at) = sL(\sin^3 at) \qquad ...(1)$

Now, $\quad \sin 3x = 3 \sin x - 4 \sin^3 x \quad \Rightarrow \quad \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$

$$\therefore \quad \sin^3 at = \frac{3}{4} \sin at - \frac{1}{4} \sin 3at$$

$$\therefore \quad L(\sin^3 at) = \frac{3}{4} L(\sin at) - \frac{1}{4} L(\sin 3at)$$

$$= \frac{3}{4}\left(\frac{a}{s^2 + a^2}\right) - \frac{1}{4}\left(\frac{3a}{s^2 + (3a)^2}\right)$$

$$= \frac{3a}{4}\left(\frac{1}{s^2 + a^2} - \frac{1}{s^2 + 9a^2}\right) = \frac{6a^3}{(s^2 + a^2)(s^2 + 9a^2)}$$

$$\therefore \quad (1) \quad \Rightarrow \quad 3a\, L(\cos at \sin^2 at) = s \cdot \frac{6a^3}{(s^2 + a^2)(s^2 + 9a^2)} - 0$$

$$\therefore \quad L(\cos at \sin^2 at) = \frac{1}{3a} \cdot \frac{6a^3 s}{(s^2 + a^2)(s^2 + 9a^2)} = \boldsymbol{\frac{2a^2 s}{(s^2 + a^2)(s^2 + 9a^2)}}.$$

Alternative method

$$\sin^2 at \cos at = \frac{1}{2} . \sin at . 2 \sin at \cos at$$

$$= \frac{1}{2} \sin at \sin 2at = \frac{1}{4} . 2 \sin 2at \sin at$$

$$= \frac{1}{4} (\cos at - \cos 3at).$$

$$\therefore \quad L(\sin^2 at \cos at) = L\left(\frac{1}{4}(\cos at - \cos 3at)\right)$$

$$= \frac{1}{4}[L(\cos at) - L(\cos 3at)] = \frac{1}{4}\left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + 9a^2}\right]$$

$$= \frac{s}{4}\left[\frac{8a^2}{(s^2 + a^2)(s^2 + 9a^2)}\right] = \boldsymbol{\frac{2a^2 s}{(s^2 + a^2)(s^2 + 9a^2)}}.$$

Theorem 2. *(Laplace transform of the derivative of any order n). Let f(t) be a real function such that*

(*i*) *$f(t), f'(t), f''(t), ..., f^{(n-1)}(t)$ are continuous for all $t \geq 0$*

(*ii*) *there exists constants k and M such that*

$| f(t) | \leq Me^{kt}$ for $t \geq 0$ and $| f^{(i)}(t) | \leq Me^{kt}$ for $t \geq 0$ and $i = 1, 2, ..., n - 1$

(*iii*) *$f^{(n)}(t)$ is piece wise continuous on every finite interval in the range $t \geq 0$ then the Laplace transform of $f^{(n)}(t)$ exists and*

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - ...... - f^{(n-1)}(0) \quad \text{for } s > k.$$

Proof. Using (*i*), $f'(t)$ is piece wise continuous on every finite interval in the range $t \geq 0$.

$\therefore$ By **theorem 1,** $L(f') = s\,L(f) - f(0),\ s > k$

By applying **theorem 1** to the function $f'(t)$, we get

$$L(f'') = s\,L(f') - f'(0).$$

$$\therefore \qquad L(f'') = s\,[s\,L(f) - f(0)] - f'(0)$$

or $$L(f'') = s^2\,L(f) - s\,f(0) - f'(0)$$

Let the result be true for $f^{(m)}(t)$, the mth derivative of $f(t)$.

$$\therefore \qquad L(f^{(m)}) = s^m\,L(f) - s^{m-1}\,f(0) - \ldots\ldots\ f^{(m-1)}(0) \qquad \ldots(1)$$

Now (*i*) $f^{(m)}(t)$ is continuous for all $t \geq 0$.

(*ii*) $|\ f^{(m)}(t)\ | \leq M\,e^{kt}$ for $t \geq 0$.

(*iii*) By given condition (*i*), $f^{(m+1)}(t)$ is piece wise continuous on every finite interval in the range $t \geq 0$.

$\therefore$ By **theorem 1**, the Laplace transform of the derivative $f^{(m+1)}(t)$ exists and

$$L(f^{(m+1)}) = s\,L(f^{(m)}) - f^{(m)}(0). \qquad \ldots(2)$$

Using (1), we get

$$L(f^{(m+1)}) = s\,[s^m\,L(f) - s^{m-1}\,f(0) - \ldots\ldots - f^{(m-1)}(0)] - f^{(m)}(0)$$

or $$L(f^{(m+1)}) = s^{m+1}\,L(f) - s^m\,f(0) - s^{m-1}\,f(0) - \ldots\ldots - f^{m}(0)$$

$\therefore$ By P.M.I., we have

$$\mathbf{L(f^{(n)}) = s^n\,L(f) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \ldots\ldots - f^{(n-1)}(0) \quad for\ s > k.}$$

In particular

$n = 1 \quad \Rightarrow \qquad \mathbf{L(f') = sL(f) - f(0)}$

$n = 2 \quad \Rightarrow \qquad \mathbf{L(f'') = s^2L(f) - sf(0) - f'(0)}$

$n = 3 \quad \Rightarrow \qquad \mathbf{L(f''') = s^3L(f) - s^2f(0) - sf'(0) - f''(0).}$

Example 3. *Given* $L(t^3) = \dfrac{6}{s^4}$, *find the value of* $L(t^6)$.

Sol. Let $\qquad f(t) = t^6.$

$\therefore \qquad f'(t) = 6t^5,\ f''(t) = 30t^4 \quad \text{and} \quad f'''(t) = 120t^3.$

We have $\qquad L(f''') = s^3L(f) - s^2\,f(0) - sf'(0) - f''(0).$

$$\therefore \qquad L(120t^3) = s^3L(t^6) - s^2\,.\,(0)^6 - s\,.\,6(0)^5 - 30(0)^4$$

$$\Rightarrow \qquad 120L(t^3) = s^3L(t^6) - 0 - 0 - 0$$

$$\Rightarrow \qquad 120\left(\frac{6}{s^4}\right) = s^3L(t^6) \qquad \text{(Given)}$$

$$\Rightarrow \qquad L(t^6) = \frac{1}{s^3}\left(\frac{720}{s^4}\right) = \mathbf{\frac{720}{s^7}}.$$

Example 4. *Find the Laplace transform of the function cos at,* $t \geq 0$.

Sol. Let $\qquad f(t) = \cos at,\ t \geq 0.$

$\therefore \qquad f'(t) = -\,a\,\sin at \quad \text{and} \quad f''(t) = -\,a^2\cos at.$

We have $\qquad L(f'') = s^2\,L(f) - sf(0) - f'(0).$

$\therefore \quad L(-a^2 \cos at) = s^2 L(\cos at) - s \cos 0 - (-a \sin 0)$

$\Rightarrow \quad -a^2 L(\cos at) = s^2 L(\cos at) - s - 0$

$\Rightarrow \quad (s^2 + a^2) L(\cos at) = s$

$$\therefore \quad L(\cos at) = \frac{\mathbf{s}}{\mathbf{s^2 + a^2}}.$$

Example 5. *Find the Laplace transform of the function t sin bt.*

Sol. Let $f(t) = t \sin bt.$

$\therefore \quad f'(t) = t(b \cos bt) + 1 . \sin bt = bt \cos bt + \sin bt$

and $\quad f''(t) = b[t(-b \sin bt) + 1 . \cos bt] + b \cos bt = -b^2 t \sin bt + 2b \cos bt$

We have $\quad L(f'') = s^2 L(f) - sf(0) - f'(0).$

$\therefore \quad L(-b^2 t \sin bt + 2b \cos bt) = s^2 L(t \sin bt) - s(0 . \sin 0) - (b . 0 . \cos 0 + \sin 0)$

$\Rightarrow \quad -b^2 L(t \sin bt) + 2b L(\cos bt) = s^2 L(t \sin bt) - 0 - (0 + 0)$

$$\Rightarrow \quad (s^2 + b^2) L(t \sin bt) = 2b L(\cos bt) = 2b \left(\frac{s}{s^2 + b^2}\right)$$

$$\Rightarrow \quad L(t \sin bt) = \frac{1}{s^2 + b^2}\left(\frac{2bs}{s^2 + b^2}\right) = \frac{\mathbf{2bs}}{\mathbf{(s^2 + b^2)^2}}.$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. $L(f') = sL(f) - f(0).$

Rule II. $L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots\ldots - f^{(n-1)}(0).$

Rule III. (*i*) $L(f'') = s^2 L(f) - sf(0) - f'(0)$

(*ii*) $L(f''') = s^3 L(f) - s^2 f(0) - sf'(0) - f''(0).$

TEST YOUR KNOWLEDGE

1. Using $L(1) = \frac{1}{s}$, find $L(t)$.

2. (*i*) Using $L(\sinh at) = \frac{a}{s^2 - a^2}$, find $L(\cosh at)$.

(*ii*) Using $L(\cosh at) = \frac{s}{s^2 - a^2}$, find $L(\sinh at)$.

3. (*i*) Using $L(\sin bt) = \frac{b}{s^2 + b^2}$, find $L(\cos bt)$.

(*ii*) Using $L(\cos bt) = \frac{s}{s^2 + b^2}$, find $L(\sin bt)$.

4. Find the Laplace transform of the function $\sin bt \cos^2 bt$, $t \geq 0$.

5. Using $L(1) = \frac{1}{s}$, find $L(t^2)$.

6. Given $L(t^2) = \frac{2}{s^3}$, find the value of $L(t^5)$.

7. Find the Laplace transform of the following functions by using the result of Laplace transform of second derivative of a function :

(*i*) e^{at} (*ii*) sinh at (*iii*) cosh at (*iv*) sin at.

8. Find the Laplace transform of the function t cos at.

Answers

1. $\dfrac{1}{s^2}$ **2.** (*i*) $\dfrac{s}{s^2-a^2}$ (*ii*) $\dfrac{a}{s^2-a^2}$ **3.** (*i*) $\dfrac{s}{s^2+b^2}$ (*ii*) $\dfrac{b}{s^2+b^2}$

4. $\dfrac{b\,(s^2+3b^2)}{(s^2+b^2)\,(s^2+9b^2)}$ **5.** $\dfrac{2}{s^3}$ **6.** $\dfrac{120}{s^6}$ **7.** (*i*) $\dfrac{1}{s-a}$

(*ii*) $\dfrac{a}{s^2-a^2}$ (*iii*) $\dfrac{s}{s^2-a^2}$ (*iv*) $\dfrac{a}{s^2+a^2}$ **8.** $\dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Hint

8. Let $f(t) = t \cos at$.
Use $L(f'') = s^2L(f) - sf(0) - f'(0)$.

1.13. LAPLACE TRANSFORMS OF INTEGRALS

Theorem. *Let f(t) be a real function such that*

(*i*) *f(t) is piece wise continuous on every finite interval in the range* $t \geq 0$

(*ii*) *there exists constants k and M such that* $|f(t)| \leq Me^{kt}$ *for* $t \geq 0$, *then the Laplace transform of the integral* $\int_0^t f(T)\,dT$ *exists and*

$$L\left(\int_0^t f(T)\,dT\right) = \frac{1}{s}\,L(f(t)) \text{ for } s > k.$$

Proof. If k in (*ii*) is negative, then $-k > 0$ and

$$(ii) \quad \Rightarrow \quad |f(t)| \leq Me^{kt} = \frac{M}{e^{-kt}} \leq Me^{-kt}. \qquad (\because \; e^{-kt} > 1)$$

$\therefore$ We can assume that k is positive.

Let $$g(t) = \int_0^t f(T)\,dT, \quad t \geq 0$$

$\therefore$ $g(t)$ is continuous for all $t \geq 0$.

Now $$|g(t)| = \left|\int_0^t f(T)\,dT\right| \leq \int_0^t |f(T)|\,dT \leq \int_0^t |Me^{kT}|\,dT = M\int_0^t e^{kT}\,dT$$

$$= \frac{M}{k}(e^{kt}-1) \leq \frac{M}{k}e^{kt}. \qquad (\because \; k > 0 \;\Rightarrow\; e^{kt} > 1)$$

$$\therefore \quad |g(t)| \leq \frac{M}{k}e^{kt} \quad \text{for } t \geq 0$$

Also $g'(t) = f(t)$ except for points at which $f(t)$ is discontinuous.

$\therefore$ $g'(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$.

$\therefore$ The Laplace transform of $g'(t)$ exists and

$$L(g') = sL(g) - g(0) \quad \text{for } s > k$$

$$\Rightarrow \quad L(g') = sL(g) \quad \text{for } s > k \qquad \left(\because \; g(0) = \int_0^0 f(T)\, dT = 0\right)$$

$$\Rightarrow \quad L(g) = \frac{1}{s} L(g')$$

$$\Rightarrow \quad \mathbf{L\left(\int_0^t f(T)\, dT \right) = \frac{1}{s} L(f(t)) \quad for\ s > k.}$$

WORKING RULE FOR SOLVING PROBLEMS

Rule $\quad L\left(\int_0^t f(T)\, dT\right) = \frac{1}{s} L(f(t))$.

ILLUSTRATIVE EXAMPLES

Example 1. *Using* $L(1) = \frac{1}{s}$, *find* $L(t^2)$.

Sol. Let $f(t) = 1$. $\quad \therefore \quad L(f) = L(1) = \frac{1}{s}$ (Given)

We have $\quad L\left(\int_0^t f(T)\, dT \right) = \frac{1}{s} L(f)$.

$$\therefore \quad L\left(\int_0^t 1\, dT \right) = \frac{1}{s}\left(\frac{1}{s}\right)$$

$$\Rightarrow \quad L\left(T \Big|_0^t \right) = \frac{1}{s^2} \;\Rightarrow\; L(t) = \frac{1}{s^2}.$$

Let $\quad g(t) = t. \quad \therefore \quad L(g) = L(t) = \frac{1}{s^2}$

We have $\quad L\left(\int_0^t g(T)\, dT \right) = \frac{1}{s} L(g)$

$$\therefore \quad L\left(\int_0^t T\, dT \right) = \frac{1}{s} \cdot \frac{1}{s^2}$$

$$\Rightarrow \quad L\left(\frac{T^2}{2} \Big|_0^t \right) = \frac{1}{s^3} \;\Rightarrow\; L\left(\frac{t^2}{2}\right) = \frac{1}{s^3}$$

$$\Rightarrow \quad \frac{1}{2} L(t^2) = \frac{1}{s^3} \;\Rightarrow\; L(t^2) = \mathbf{\frac{2}{s^3}}.$$

Example 2. *Using* $L\,(\sin bt) = \dfrac{b}{s^2 + b^2}$ *, find L (cos bt).*

Sol. Let $f(t) = \sin bt$. $\therefore$ $L(f) = L\,(\sin bt) = \dfrac{b}{s^2 + b^2}$. (Given)

We have $$L\left(\int_0^t f(T)\,dT\right) = \frac{1}{s}L(f).$$

$$\therefore \quad L\left(\int_0^t \sin bT\,dT\right) = \frac{1}{s}\cdot\frac{b}{s^2+b^2} \quad \Rightarrow \quad L\left(-\left.\frac{\cos bT}{b}\right|_0^t\right) = \frac{b}{s\,(s^2+b^2)}$$

$$\Rightarrow \quad L\left(\frac{1}{b} - \frac{\cos bT}{b}\right) = \frac{b}{s\,(s^2+b^2)} \quad \Rightarrow \quad \frac{1}{b}L(1) - \frac{1}{b}L\,(\cos bt) = \frac{b}{s\,(s^2+b^2)}$$

$$\Rightarrow \quad \frac{1}{b}L(\cos bt) = \frac{1}{b}\cdot\frac{1}{s} - \frac{b}{s\,(s^2+b^2)} = \frac{s}{b\,(s^2+b^2)}$$

$$\therefore \quad L(\cos bt) = \frac{\mathbf{s}}{\mathbf{s^2 + b^2}}.$$

Example 3. *Using* $L(1) = \dfrac{1}{s}$, *find the Laplace transform of the function cos bt.*

Sol. Let $f(t) = \cos bt$.

We have $$L\left(\int_0^t f(T)\,dT\right) = \frac{1}{s}L(f).$$

$$\Rightarrow \quad L\left(\int_0^t \cos bT\,dT\right) = \frac{1}{s}L(\cos bt) \quad \Rightarrow \quad L\left(\left.\frac{\sin bT}{b}\right|_0^t\right) = \frac{1}{s}L(\cos bt)$$

$$\Rightarrow \quad L\left(\frac{\sin bt}{b} - 0\right) = \frac{1}{s}L(\cos bt) \quad \Rightarrow \quad L(\sin bt) = \frac{b}{s}L(\cos bt) \qquad ...(1)$$

Let $g(t) = \sin bt$.

We have $$L\left(\int_0^t g(T)\,dT\right) = \frac{1}{s}L(g).$$

$$\Rightarrow \quad L\left(\int_0^t \sin bT\,dT\right) = \frac{1}{s}L(\sin bt) = \frac{1}{s}\cdot\left(\frac{b}{s}L(\cos bt)\right) \qquad \text{(By using (1))}$$

$$\Rightarrow \quad L\left(-\left.\frac{\cos bT}{b}\right|_0^t\right) = \frac{b}{s^2}L(\cos bt) \quad \Rightarrow \quad L\left(-\frac{\cos bt}{b} + \frac{1}{b}\right) = \frac{b}{s^2}L(\cos bt)$$

$$\Rightarrow \quad -\frac{1}{b}L(\cos bt) + \frac{1}{b}L(1) = \frac{b}{s^2}L(\cos bt) \quad \Rightarrow \quad \frac{1}{b}\cdot\frac{1}{s} = \left(\frac{b}{s^2} + \frac{1}{b}\right)L(\cos bt)$$

$$\Rightarrow \quad \frac{1}{bs} = \frac{b^2+s^2}{bs^2}L(\cos bt) \quad \Rightarrow \quad L(\cos bt) = \frac{\mathbf{s}}{\mathbf{s^2 + b^2}}.$$