Discrete Structures

Isomporphism and Planar

Introduction

•A graph can exist in different forms having the same number of vertices, edges, and also the same edge connectivity. Such graphs are called isomorphic graphs.

If we are given two simple graphs, G and H. Graphs G and H are isomorphic if there is a structure that preserves a one-to-one correspondence between the vertices and edges.

In other words, the two graphs differ only by the names of the edges and vertices but are structurally equivalent.

Introduction

- **Definition:** Let $G = (V, E)$ be a simple graph with $|V| = n$. Suppose that the vertices of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$.
- The **adjacency matrix** A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its (i, j) entry when v_i and v_j are adjacent, and 0 otherwise.
- In other words, for an adjacency matrix $A = [a_{ij}]$,
- $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G, $a_{ij} = 0$ otherwise.

Isomorphism

- **Definition:** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection (an one-to-one and onto function) f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V_1.
- Such a function f is called an **isomorphism**.
- In other words, G_1 and G_2 are isomorphic if their vertices can be ordered in such a way that the adjacency matrices M_{G_1} and M_{G_2} are identical.

Isomorphism

•From a visual standpoint, G_1 and G_2 are isomorphic if they can be arranged in such a way that their **displays are identical** (of course without changing adjacency).

•Unfortunately, for two simple graphs, each with n vertices, there are **n! possible isomorphisms proprities** that we have to check in order to show that these graphs are isomorphic.

•However, showing that two graphs are **not** isomorphic can be easy.

Isomorphism

•For this purpose we can check **invariants**, that is, properties that two isomorphic simple graphs must both hav

•They must have the follwing

• The same number of vertices,

• The same number of edges,

• The same degrees of vertices,

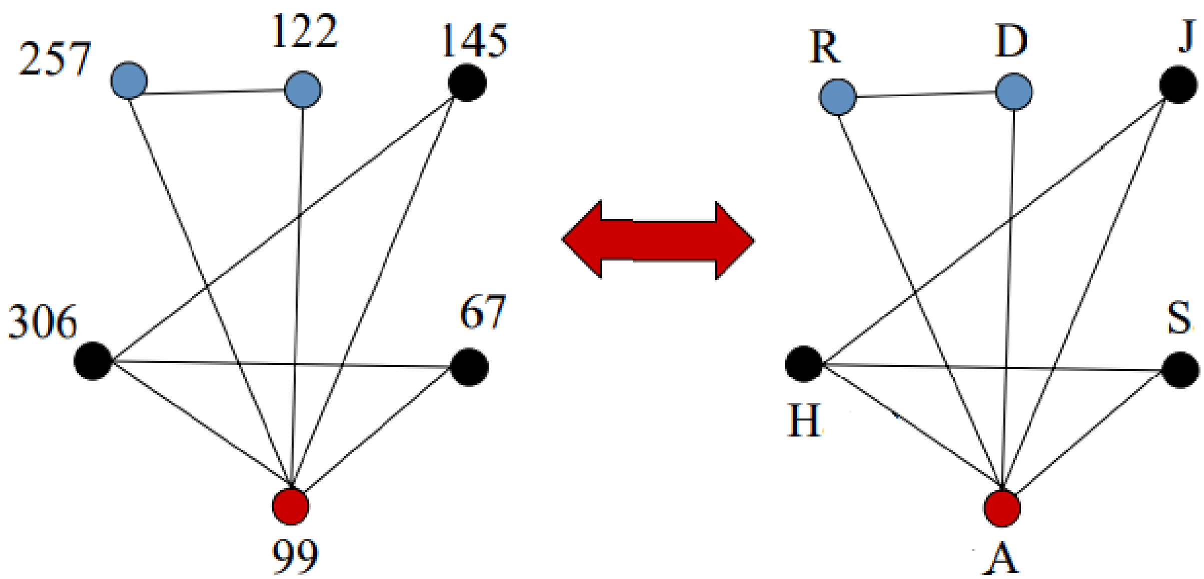
• One-to-one correspondence mapping,

• Edge preserving property and

• Adjacency matrix

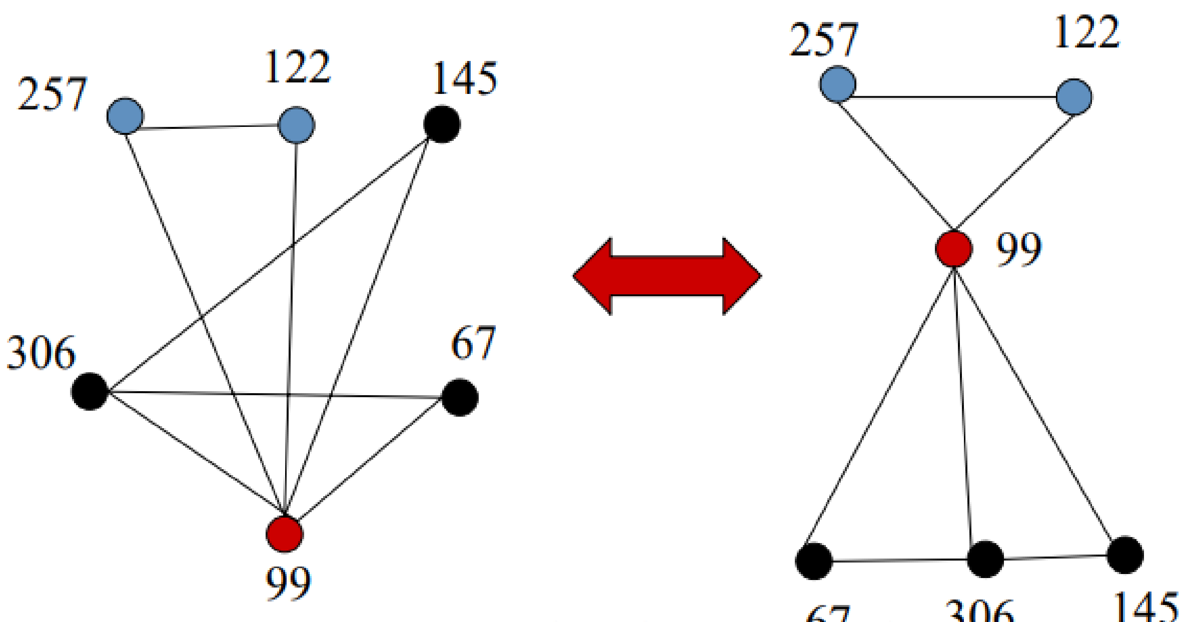
Isomorphism

Note that two graphs that **differ** in any of these invariants are not isomorphic, but two graphs that **match** in all of will achieve isomorphic.



"same" graph, different labels

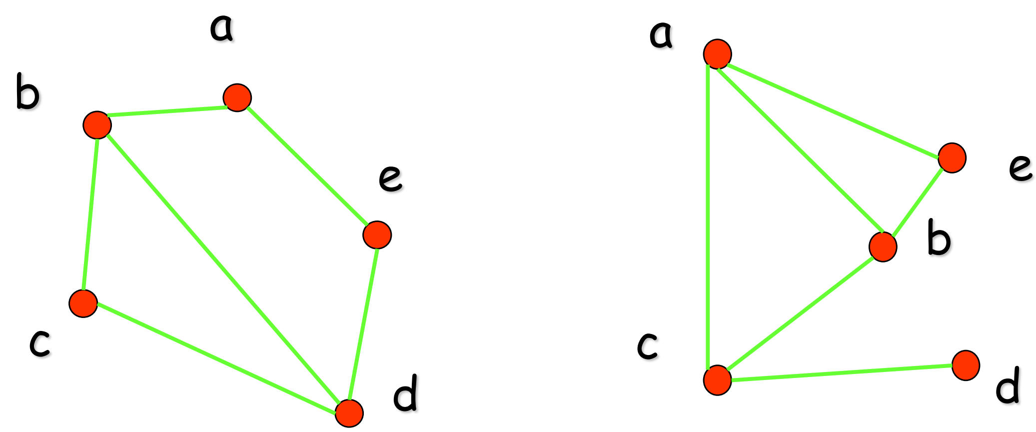
Isomorphism



Same graph, different layouts

Isomorphism

- Ex: Are the following two graphs isomorphic?



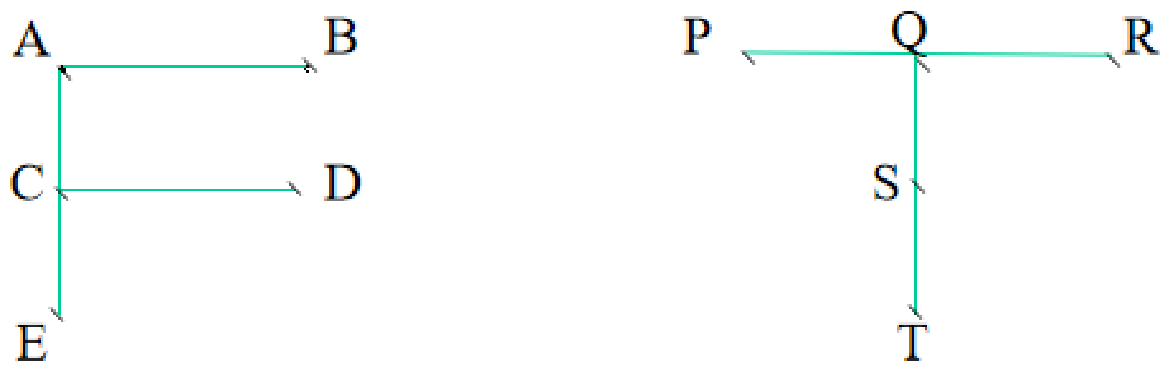
Isomorphism

•**Solution:** No, they are not isomorphic, because they differ in the degrees of their vertices.

•Vertex d in right graph is of degree one, but there is no such vertex in the left graph.

Isomorphism

Ex: Check wether the followung graphs are isomorphic or not.



G1 G2

Isomorphism

1- Number of vertices:

G1= 5 G2 = 5

2- Number of Edges:

G1 = 4 G2 = 4

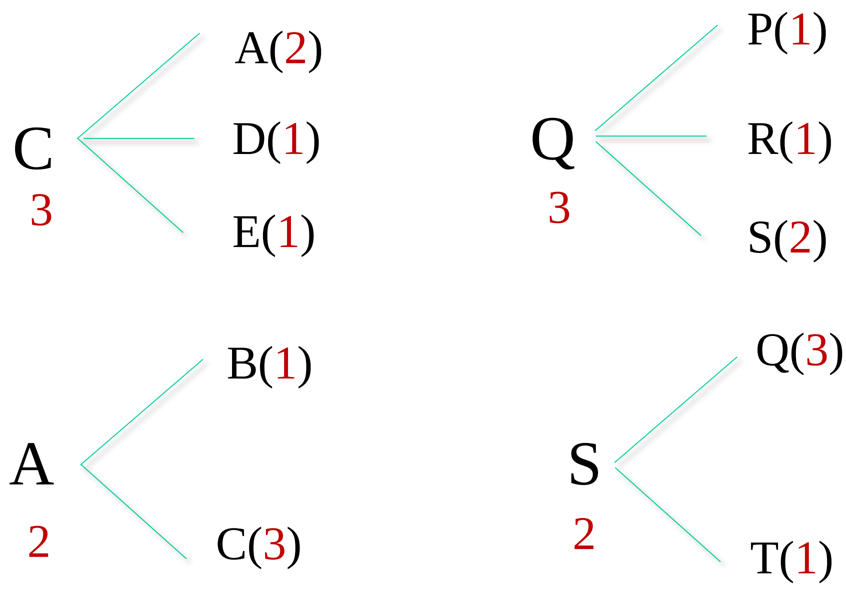
3- Degree sequence:

G1(A,B,C,D,E) = (2,1,3,1,1)

G2(P,Q,R,S,T) = (1,3,1,2,1)

Isomorphism

4- One-to-one correspondence mapping



A ⟺ S
B ⟺ T
C ⟺ Q
D ⟺ R
E ⟺ P

one-to-one correspondence between G1 and G2

Isomorphism

5- Edge preserving property:

For G1

A — B ⟺ S — T

A — C ⟺ S — Q

C — D ⟺ Q — R

C — E ⟺ Q — P

For G2

P — Q ⟺ E — C

Q — R ⟺ C — D

Q — S ⟺ C — A

S — T ⟺ A — B

Isomorphism

6- Adjacency matrix:

	A	B	C	D	E
A	0	1	1	0	0
B	1	0	0	0	0
C	1	0	0	1	1
D	0	0	1	0	0
E	0	0	1	0	0

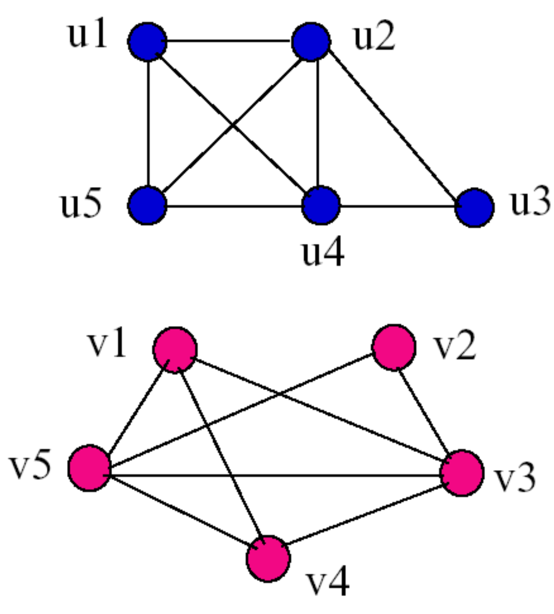
	S	T	Q	R	P
S	0	1	1	0	0
T	1	0	0	0	0
Q	1	0	0	1	1
R	0	0	1	0	0
P	0	0	1	0	0

So, G1 & G2 are isomprphic graphs

Isomorphism

- Determine if the following two graphs G_1 and G_2 are isomorphic:

(Homework)



Isomorphism

•***Note***

•If G1 ≡ G2 then −

•|V(G1)| = |V(G2)|

•|E(G1)| = |E(G2)|

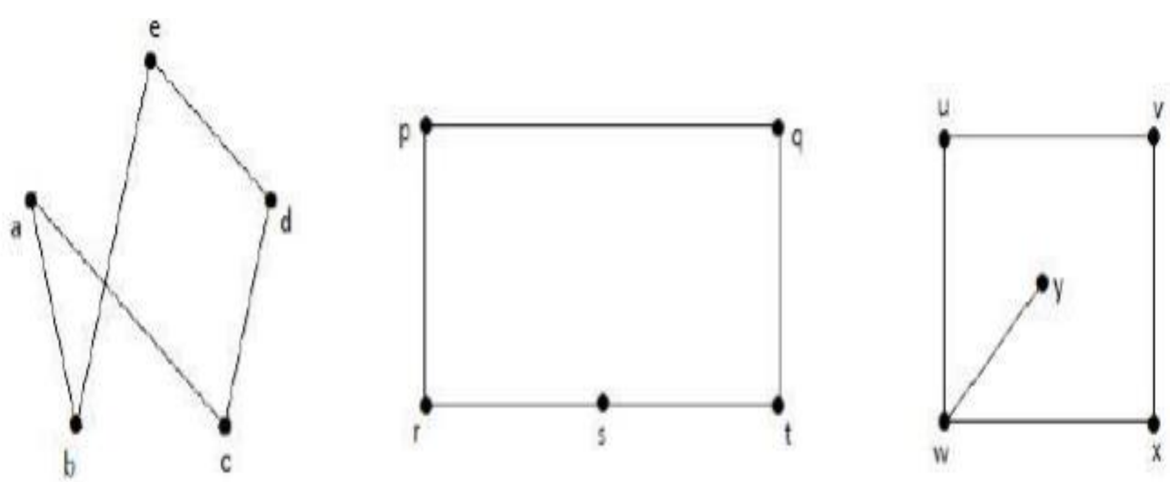
•Degree sequences of G1 and G2 are same.

All the above conditions are necessary for the graphs G1 and G2 to be isomorphic

Isomorphism

•**Example:**

•Which of the following graphs are isomorphic?



In the graph G3, vertex ‘w’ has only degree 3, whereas all the other graph vertices has degree 2.

•Hence G3 not isomorphic to G1 or G2.

Planar

Definition:

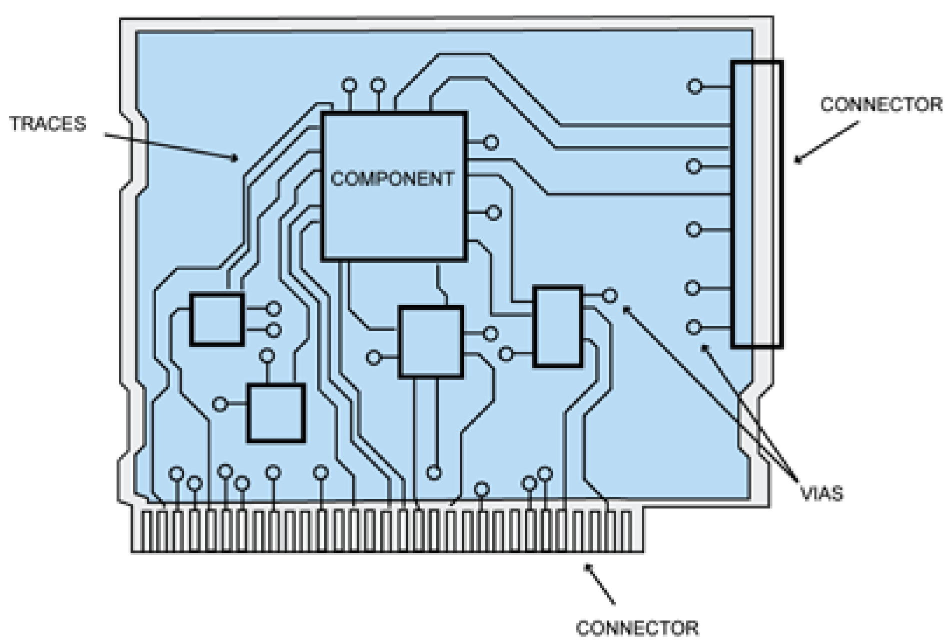
A graph that can be drawn in the plane without any of its edges intersecting is called a planar graph. A graph that is so drawn in the plane is also said to be embedded in the plane.

Applications:

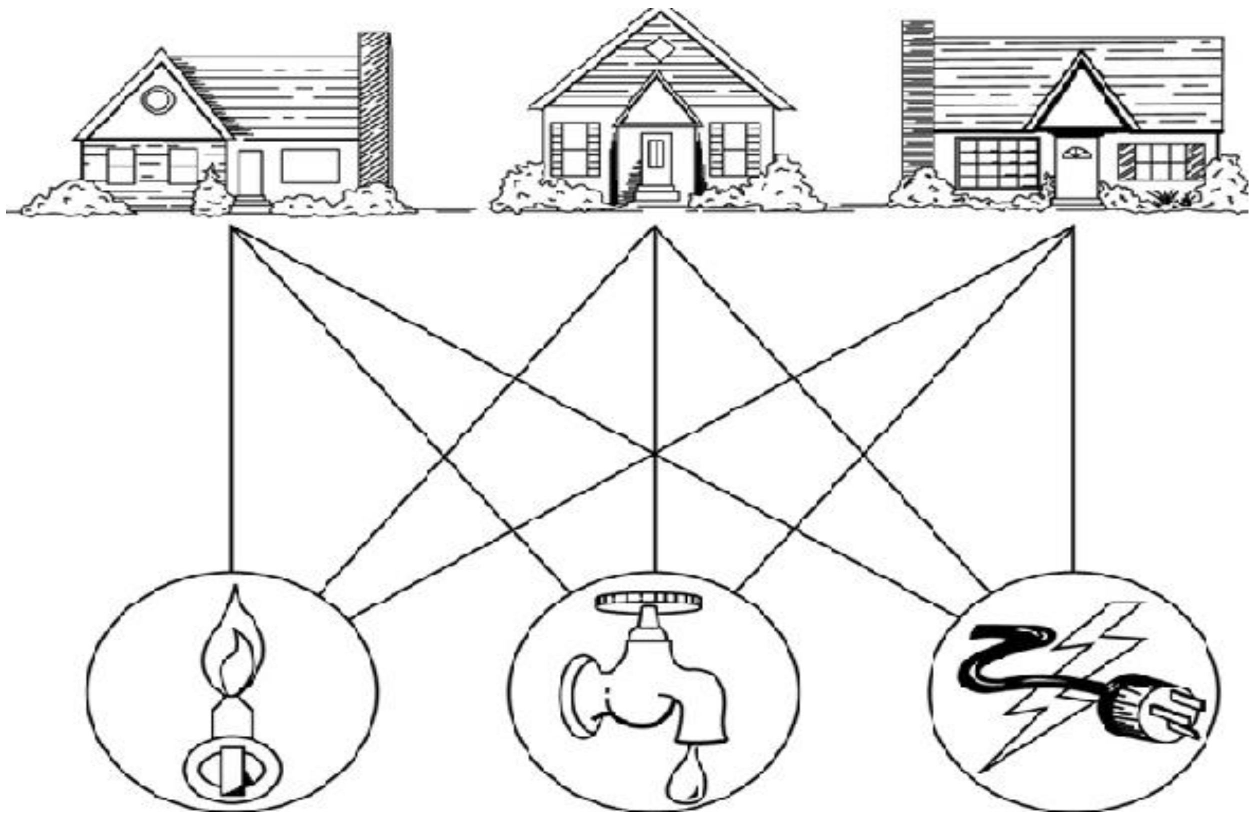
(1) circuit layout problems

(2) Three house and three utilities problem

Planar

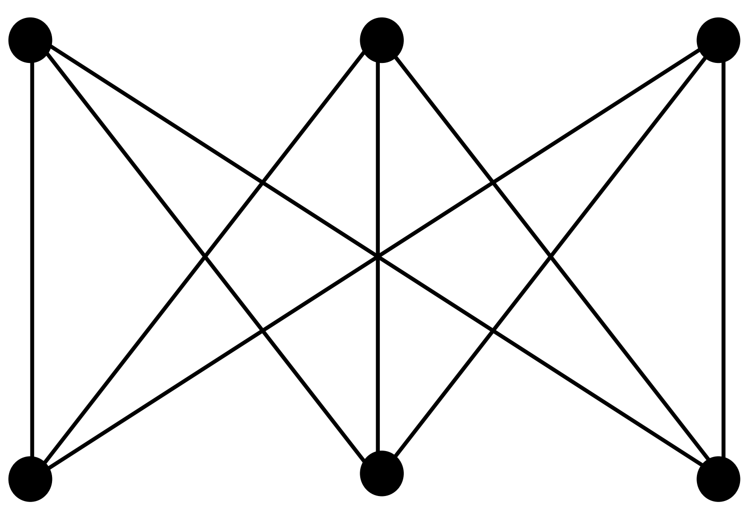


The House-and-Utilities Problem



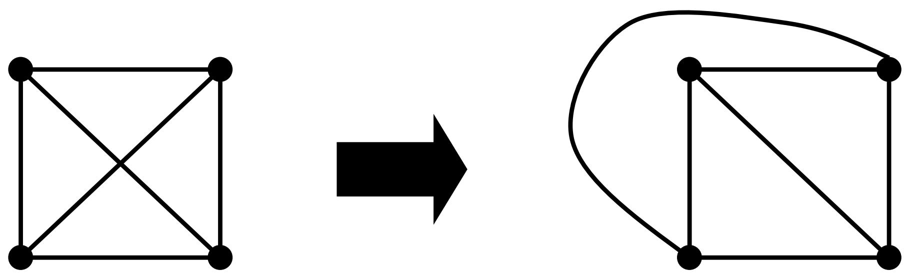
Planar

Consider the previous slide. Is it possible to join the three houses to the three utilities in such a way that none of the connections cross?



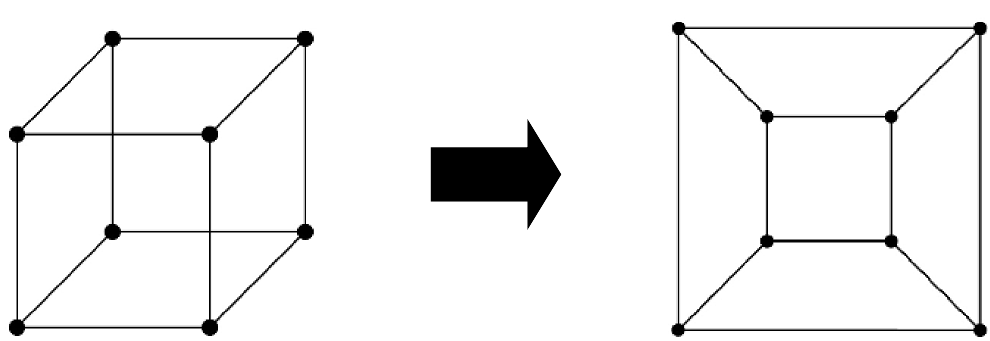
Planar

- A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in another way without crossings.



Planar

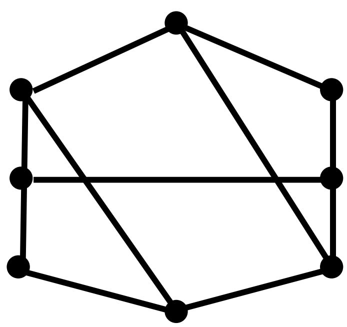
- A graph may be planar even if it represents a 3-dimensional object.



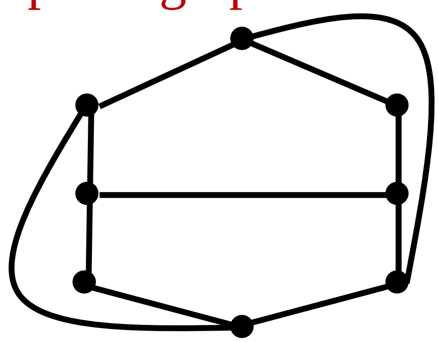
Properties of Planar Graphs

Definition:

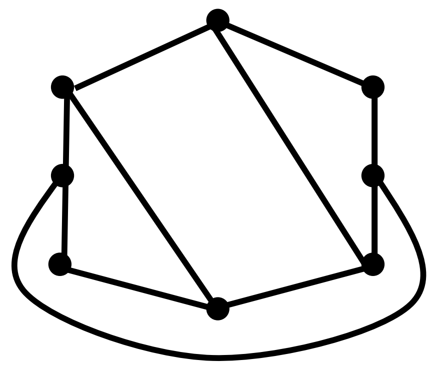
A planar graph *G* that is drawn in the plane so that no two edges intersect (that is, *G* is embedded in the plane) is called a plane graph.



(a) planar,
not a plane graph



(b) a plane graph



(c) another
plane graph

Planar

•Note. A given planar graph can give rise to several different plane graph.

Definition:

Let ***G*** be a plane graph. The connected pieces of the plane that remain when the vertices and edges of ***G*** are removed are called the regions of ***G***.

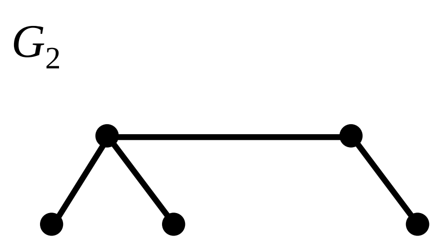
Planar

Extracted regions:
1- R1
2- R2
3- R3 (exterior)

Planar

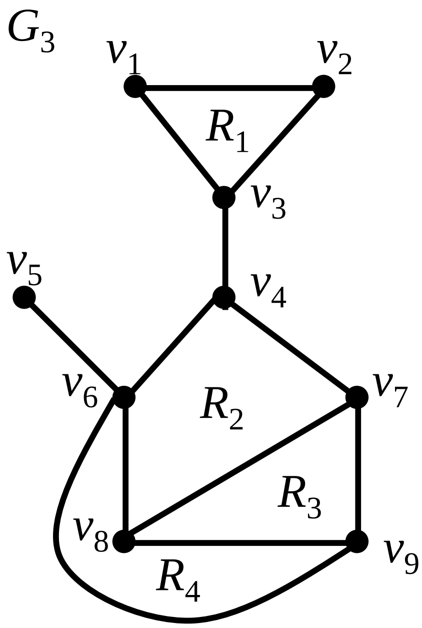
Definition:

Every plane graph has exactly one unbounded region, called the exterior region. The vertices and edges of G that are incident with a region R form a subgraph of G called the boundary of R.



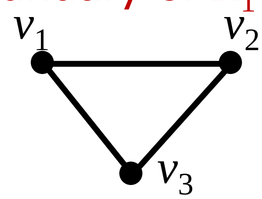
G_2 has only 1 region.

Planar



G_3 has 6 regions.

Boundary of R_1:



Boundary of R_5: