The Directional Derivative and the Gradient
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Definition: If f is a function of x & y, then the directional derivative of f in the direction $u = u_1 i + u_2 j$ at a point (x_0, y_0) is denoted by $D_u f(x_0, y_0)$ and defined by

$$\boxed{D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2}$$

If $u = cos\theta i + sin\theta j$, then

$$D_u f(x_0, y_0) = f_x(x_0, y_0)cos\theta + f_y(x_0, y_0)sin\theta$$

Where $u = \frac{\vec{u}}{|u|} = \frac{cos\theta i + sin\theta j}{\sqrt{(cos\theta)^2 + (sin\theta)^2}} = cos\theta i + sin\theta j = u_1 i + u_2 j$

$$\vec{A} = ai + bj + ck$$

$$u = \frac{\vec{A}}{|A|} = \frac{ai + bj + ck}{\sqrt{a^2 + b^2 + c^2}} = u_1 i + u_2 j + u_3 k$$

Note

Example 1: Find the directional derivative of $f(x, y) = e^{xy}$ at a point $(-2,0)$ in the direction of the unit vector $u = cos\theta i + sin\theta j$ that makes an angle of $(\frac{\pi}{3})$ with positive x-axis.

Solution:

$$f_x(x, y) = ye^{xy} \quad , \quad f_y(x, y) = xe^{xy}$$

$$f_x(-2,0) = 0 \quad , \quad f_y(-2,0) = -2$$

And $\quad u_1 = cos\theta \quad , \quad u_2 = sin\theta$

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

$$\therefore D_u f(-2,0) = 0.cos\frac{\pi}{3} + (-2)sin\frac{\pi}{3} = 0 - 2.\frac{\sqrt{3}}{2} = -\sqrt{3}$$

Example 2: Find the directional derivative of $f(x,y) = e^{xy}$ in the direction $\vec{A} = -3i + 4j$.

Solution:

$$f_x(x,y) = ye^{xy} \quad , \quad f_y(x,y) = xe^{xy}$$

$$u = \frac{-3i+4j}{\sqrt{3^2+4^2}} = \frac{-3i+4j}{5} = \frac{-3}{5}i + \frac{4}{5}j$$

$$u_1 = \frac{-3}{5} \quad , \quad u_2 = \frac{4}{5}$$

$$\therefore D_u f = f_x u_1 + f_y u_2 \Rrightarrow D_u f = (\frac{-3}{5}) ye^{xy} + (\frac{4}{5}) xe^{xy}$$

Example 3: Find the directional derivative of the function $f(x,y,z) = cosx + 2xz + y^2 - 5$ in the direction $\vec{A} = 2i + j - 2k$ at a point $\left(\frac{\pi}{2}, 0{,}1\right)$.

Solution:

$$f_x(x,y,z) = -sinx + 2z \quad \Rrightarrow \quad f_x\left(\frac{\pi}{2}, 0{,}1\right) = -sin\frac{\pi}{2}, +2(1) = 1$$

$$f_y(x,y,z) = 2y \quad \Rrightarrow \quad f_y\left(\frac{\pi}{2}, 0{,}1\right) = 2(0) = 0$$

$$f_z(x,y,z) = 2x \quad \Rrightarrow \quad f_z\left(\frac{\pi}{2}, 0{,}1\right) = 2\left(\frac{\pi}{2}\right) = \pi$$

$$\mathrm{u} = \frac{\vec{\mathrm{A}}}{|\mathrm{A}|} = \frac{2\mathrm{i}+\mathrm{j}-2\mathrm{k}}{\sqrt{2^2+1^2+(2)^2}} = \frac{2\mathrm{i}+\mathrm{j}-2\mathrm{k}}{\sqrt{9}} = \frac{2}{3}\mathrm{i} + \frac{1}{3}\mathrm{j} + \frac{(-2)}{3}\mathrm{k}$$

$$\therefore D_u f\left(\frac{\pi}{2}, 0{,}1\right) = 1\left(\frac{2}{3}\right) + 0\left(\frac{1}{3}\right) + \pi\left(\frac{-2}{3}\right) = \frac{2}{3} - \frac{2}{3}\pi$$

Definition: If f is a function of x & y, then the gradient of f is defined by

$$\boxed{\nabla f(x_0,y_0) = f_x(x_0,y_0)i + f_y(x_0,y_0)j}$$

And relation between gradient and directional derivative

$$D_u f(x_0,y_0) = u.\nabla f(x_0,y_0)$$

If f is a function of x,y & z, then the gradient of f is defined by

$$\boxed{\nabla f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)i + f_y(x_0,y_0,z_0)j + f_z(x_0,y_0,z_0)k}$$

Example 4: Let $f(x,y,z)=x^2+y^2+2xz$, find $\nabla f(1,0,2)$.

Solution:

$$f_x(x,y,z)=2x+2z \quad \Rrightarrow \quad f_x(1,0,2)=6$$
$$f_y(x,y,z)=2y \quad \Rrightarrow \quad f_y(1,0,2)=0$$
$$f_z(x,y,z)=2x \quad \Rrightarrow \quad f_z(1,0,2)=2$$
$$\therefore \nabla f(1,0,2)=6i+2k$$

Example 5: Find the gradient of the function

$f(x,y,z)=e^{xy}-2cosz+x^2z$ at the point (2,1,5).

Solution:

$$f_x(x,y,z)=ye^{xy}+2xz \quad \Rrightarrow \quad f_x(2,1,5)=e^2+20$$
$$f_y(x,y,z)=xe^{xy} \quad \Rrightarrow \quad f_y(2,1,5)=2(e^2)$$
$$f_z(x,y,z)=2sinz+x^2 \quad \Rrightarrow \quad f_z(2,1,5)=2sin5+4$$
$$\nabla f(2,1,5)=(e^2+20)i+2(e^2)j+(2sin5+4)k$$

Example 6: Find the directional derivative and gradient of the function $f(x,y,z)=e^{xy}-sinxz$ in the direction $\vec{A}=i-j+k$ at a point (2,1,0) .

Solution:

$$f_x(x,y,z)=ye^{xy}-zcosxz \quad \Rrightarrow \quad f_x(2,1,0)=e^2$$
$$f_y(x,y,z)=xe^{xy} \quad \Rrightarrow \quad f_y(2,1,0)=2(0)=2e^2$$
$$f_z(x,y,z)=-xcosxz \quad \Rrightarrow \quad f_z(2,1,0)=-2$$

$$\mathrm{u}=\frac{\vec{\mathrm{A}}}{|\mathrm{A}|}=\frac{\mathrm{i}-\mathrm{j}+\mathrm{k}}{\sqrt{1^2+(-1)^2\ +(1)^2}}=\frac{\mathrm{i}-\mathrm{j}+\mathrm{k}}{\sqrt{3}}=\frac{1}{\sqrt{3}}\mathrm{i}-\frac{1}{\sqrt{3}}\mathrm{j}+\frac{1}{\sqrt{3}}\mathrm{k}$$

$$\therefore D_uf(2,1,0)=e^2\left(\frac{1}{\sqrt{3}}\right)-2e^2\left(\frac{1}{\sqrt{3}}\right)-2\left(\frac{1}{\sqrt{3}}\right)$$

And

$$\nabla f(2,1,0)=e^2i+2e^2j-2k$$