## *4th Lecture*

## Closure of a Set انغلاق المجموعة

**Definition (1.5):** Let $(X, \tau)$ be a topological space and $E \subset X$, we define the **closure** of a set $E$, denoted by $\bar{E}$ as follows:

$$\bar{E} = \bigcap_{\forall F \supset E} F, \text{ where } F \text{ is closed}$$

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**Example (1.10):**

Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}, X\}$.

Let $E = \{d, e\}$, $G = \{a, e\}$ and $H = \{d\}$. Find $\bar{E}$, $\bar{G}$ and $\bar{H}$.

**Solution:**

The closed sets in $(X, \tau)$ are $X$, $\{b, c, d, e\}$, $\{a, d, e\}$, $\{d, e\}$, $\{a\}$, $\emptyset$

$\bar{E} = \bigcap_{\forall F \supset E} F$ , where $F$ is closed

$= X \cap \{b, c, d, e\} \cap \{a, d, e\} \cap \{d, e\} = \{d, e\}$

$\bar{G} = \bigcap_{\forall F \supset G} F$ , where $F$ is closed

$= X \cap \{a, d, e\} = \{a, d, e\}$

$\bar{H} = \bigcap_{\forall F \supset H} F$ , where $F$ is closed

$= X \cap \{a, d, e\} \cap \{b, c, d, e\} \cap \{d, e\} = \{d, e\}$

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**Remark (1.6):**

(1) Since $\bigcap_{\forall F \supset E} F \subset F$, $\forall\ closed\ F$. Thus $\bar{E}$ is the smallest closed set containing $E$.

(2) If $E$ is itself closed then $\bar{E} = E$.

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**Theorem (1.4):** If $E$ is a subset of $(X,\tau)$.Then

$$\bar{E} = E \cup d(E)$$

**Proof:**

We have

$E \subset \bar{E}$ and $d(E) \subset \bar{E}$

$\Rightarrow E \cup d(E) \subset \bar{E}$ .................. (1)

We need to show that $\bar{E} \subset E \cup d(E)$

Let $x \notin E \cup d(E)$

$\Rightarrow x \notin E \wedge x \notin d(E)$

Since $x \notin d(E)$

$\Rightarrow \exists\ open\ G_x \ni x, (E \cap G_x) \backslash \{x\} = \emptyset$

$\Rightarrow \forall\ x \notin E \cup d(E), \exists\ open\ G_x \ni x, (E \cap G_x) \backslash \{x\} = \emptyset$

$\Rightarrow \forall\ x \notin E \cup d(E), x \notin \bar{E}$

$\Rightarrow \bar{E} \subset E \cup d(E)$ .................. (2)

From (1) and (2) we get

$$\bar{E} = E \cup d(E)$$

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**Theorem (1.5): (Closure Axioms)** بديهيات الانغلاق

If $A, B$ are subsets of a topological space $(X,\tau)$ then

(i) $\bar{\emptyset} = \emptyset \wedge \bar{X} = X$.

(ii) $A \subset \bar{A}$.

(iii) $\overline{(\bar{A})} = \bar{A}$.

(iv) $A \subset B \Rightarrow \bar{A} \subset \bar{B}$.

(v) $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.

**Proof:**

**(i)** Since $\emptyset$ is closed $\Rightarrow \overline{\emptyset} = \emptyset$.

Also $X$ is closed $\Rightarrow \bar{X} = X$.

**(ii)** We have $\bar{A} = \bigcap_{F \ closed} F, \ \forall \ F \supset A$

$\Rightarrow \bigcap_{F \ closed} F \supset A \Rightarrow \bar{A} \supset A$ or $A \subset \bar{A}$

**(iii)** Let $E = \bar{A}$ , since $E$ is closed $\Rightarrow \bar{E} = E$

$\Rightarrow \overline{(\bar{E})} = \bar{E}$

**(iv)** We have $\bar{A} = \bigcap_{\forall F \supset A} F$ and $\bar{B} = \bigcap_{\forall F \supset B} F$

Since $A \subset B$ then we have

$\bar{B} = \bigcap_{\forall F \supset B \subset A} F \supset \bigcap_{\forall F \supset A} F = \bar{A}$

$\Rightarrow \bar{A} \subset \bar{B}$

**(v)** $\overline{(A \cup B)} = (A \cup B) \cup d(A \cup B)$

$= A \cup B \cup d(A) \cup d(B)$

$= (A \cup d(A)) \cup (B \cup d(B))$

$= \bar{A} \cup \bar{B}$

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**Exercises (1.4): (Homework)**

(1) Disprove that $\overline{(A \cap B)} = \bar{A} \cap \bar{B}$. (Give an example)

(2) Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\}$

Let $E = \{a, d\}$, $G = \{b, d\}$ and $H = \{b, c\}$. Find $\bar{E}$, $\bar{G}$ and $\bar{H}$.

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