B) If the denominator is decomposed into roots or first-order factors, some of which are repeated

denominator: $(x+a)(x+b)^2$

$(x+a)(x+b)^3$

$(x+a)^4(x+b)$

and so on

The integration is as follows:-

1. $\int \frac{A}{x+a}\,dx + \int \frac{B}{x+b}\,dx + \int \frac{C}{(x+b)^2}$

2. $\int \frac{A}{x+a}\,dx + \int \frac{B}{(x+b)}\,dx + \int \frac{C}{(x+b)^2}\,dx + \int \frac{D}{(x+b)^3}\,dx$

3. $\int \frac{A}{x+a}\,dx + \int \frac{B}{(x+a)^2}\,dx + \int \frac{C}{(x+a)^3}\,dx + \int \frac{D}{(x+a)^4}\,dx + \int \frac{E}{x+b}\,dx$

Ex

Evaluate as the following:-

$$\int \frac{x}{(x-1)^2}\,dx$$

$$\int \frac{x}{(x-1)^2}\,dx = \int \frac{A}{(x-1)}\,dx + \int \frac{B}{(x-1)^2}\,dx$$

$$\left[\frac{x}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}\right] * (x-1)^2$$

$$X = A(X-1) + B$$
$$X = AX - A + B$$

$$X = AX \Rightarrow A = 1 \quad ①$$

$$B - A = 0 \Rightarrow B = A = 1 \quad ②$$

$$\therefore \int \frac{X}{(X-1)^2} dx = \int \frac{1}{X-1} dx + \int \frac{1}{(X-1)^2} dx$$

$$= \ln|X-1| + \int (X-1)^{-2} dx$$

$$= \ln|X-1| + \frac{(X-1)^{-1}}{-1}$$

$$= \ln|X-1| - \frac{1}{(X-1)} + c$$

2. $$\int \frac{X+2}{(X-1)(X-2)^2} dx$$

$$\int \frac{X+2}{(X-1)(X-2)^2} dx = \int \frac{A}{X-1} dx + \int \frac{B}{X-2} dx + \int \frac{C}{(X-2)^2} dx$$

$$\left[\frac{X+2}{(X-1)(X-2)^2} = \frac{A}{X-1} + \frac{B}{X-2} + \frac{C}{(X-2)^2} \right] * (X-1)(X-2)^2$$

$$X + 2 = A(X-2)^2 + B(X-1)(X-2) + C(X-1)$$
$$X + 2 = AX^2 - 4AX + 4A + BX^2 - 3BX + 2B + CX - C$$

$$AX^2 + BX^2 = 0 \quad ① \quad \div x^2 \Rightarrow A = -B$$
$$-4AX - 3BX + CX = X \quad ② \quad \div x \Rightarrow -4A - 3B + C = 1$$
$$4A + 2B - C = 2 \quad ③$$

Adding the two equations ② and ③ we get:

$$-4A - 3B + C = 1$$
$$4A + 2B - C = 2$$

$$-B = 3 \Rightarrow B = -3$$

From equation ① we get $A = -B \Rightarrow A = 3$

Substituting the values of A and B into equation 3, we get:-

$$4A + 2B - C = 2$$
$$12 - 6 - C = 2 \Rightarrow C = 4$$

$$\therefore \int \frac{x+2}{(x-1)(x-2)^2}\,dx = \int \frac{3}{x-1}\,dx - \int \frac{3}{x-2}\,dx + \int \frac{4}{(x-2)^2}\,dx$$

$$= 3\ln|x-1| - 3\ln|x-2| + 4\int (x-2)^{-2}\,dx$$

$$= 3\ln|x-1| - 3\ln|x-2| - \frac{4}{x-2} + c$$

H.w.

1. $$\int \frac{x+2}{x(x^2+2x+1)}\,dx$$

$$x(x^2+2x+1) = x(x+1)(x+1)^2$$

2. $$\int \frac{2x^2-3x+3}{x^3-2x^2+x} \Rightarrow x(x^2-2x+1) \Rightarrow x(x-1)^2$$

3. $$\int \frac{xe^x}{(x+1)^2}\,dx$$

$$u = xe^x \;;\; dv = (x+1)^{-2}\,dx$$

C) If some roots are of the second degree and not analytical, then the numerator is straight linear equation:

$$\int \text{البسط} = \int \frac{Ax+B}{x^2+a}$$

For example

1. $\displaystyle\int \frac{x^2+2x}{x(x^2+1)}\,dx$

$$\int \frac{x^2+2x}{x(x^2+1)}\,dx = \int \frac{A}{x}\,dx + \int \frac{Bx+C}{x^2+1}\,dx$$

$$\left(\frac{x^2+2x}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}\right) * x(x^2+1)$$

$$x^2+2x = A(x^2+1)+(Bx+C)x$$

$$x^2+2x = Ax^2+A+Bx^2+Cx$$

$x^2 = Ax^2+Bx^2$ ① $\div x^2 \Rightarrow 1 = A+B$

$2x = Cx$ ② $\div x \Rightarrow C = 2$

$0 = A$ ③ $\Rightarrow B = 1$

$$\therefore \int \frac{x^2+2x}{x(x^2+1)}\,dx = \int \frac{x+2}{x^2+2}\,dx$$

$$= \int \frac{x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$$= 2\ln|x^2+1| + 2\tan^{-1}x + c$$

2. $\int \frac{1}{x^2(x^2+4)} dx$; (x^2+4) is not able analytical

$\therefore \int \frac{1}{x^2(x^2+4)} dx = \int \frac{A}{x} dx + \int \frac{B}{x^2} dx + \int \frac{Cx+D}{x^2+4} dx$

$\left(\frac{1}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} \right) * x^2(x^2+4)$

$1 = Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2$

$1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Dx^2$

$Ax^3 + Cx^3 = 0$ (1) $\Rightarrow A = -C$

$Bx^2 + Dx^2 = 0$ (2) $\Rightarrow B = -D$

$4Ax = 0$ (3) $\Rightarrow A = 0$

$4B = 1$ (4) $\Rightarrow B = \frac{1}{4}$; $D = -\frac{1}{4}$

$\therefore \int \frac{1}{x^2(x^2+4)} dx = \int \frac{\frac{1}{4}}{x^2} dx - \int \frac{\frac{1}{4}}{x^2+4} dx$

$= \frac{1}{4} \int x^{-2} dx - \frac{1}{4} \int \frac{1}{x^2+4} dx$

$= \frac{1}{4} \frac{1}{x} - \frac{1}{4} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + c$

$= \frac{1}{4x} - \frac{1}{8} \tan^{-1} \frac{x}{2} + c$

D) Integration of function contaning roots

It is done by imposing the quantity that contain a root with any symbol and completing the integratio as follows:-

Ex

1. $$\int \frac{1}{1+\sqrt{x}}\,dx$$

let $y=\sqrt{x} \Rightarrow y^2 = x \Rightarrow dx = 2y\,dy$

$$\therefore \int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+y}\,2y\,dy$$

$$= 2\int \frac{y}{1+y}\,dy$$

Since the order of the numerator is equal to the order of the denominator, we use long division as follows:

$$\begin{array}{r} 1 \\ 1+y\,\overline{)\,y} \\ \mp y \mp 1 \\ \hline -1 \end{array}$$

$$\therefore \int \frac{1}{1+\sqrt{x}}\,dx = 2\int \frac{y}{1+y}\,dy$$

$$= 2\left[\int 1\,dy - \int \frac{1}{1+y}\,dy\right]$$

$$= 2(y - \ln|1+y|) + c$$

$$= 2\sqrt{x} - 2\ln|1+\sqrt{x}| + c$$

2. $\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx$

let $y=\sqrt{x} \Rightarrow y^2 = x \Rightarrow dx = 2y\,dy$

$\therefore \int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx = \int \frac{1+y}{1-y}\,2y\,dy$

$= \int \frac{2y+2y^2}{1-y}\,dy$

$= 2\left[\int (y+2)\,dy + \int \frac{2}{1-y}\,dy\right]$

$= 2\left(\frac{y^2}{2} + 2y + 2\ln|1-y|\right)$

$= y^2 + 4y + 4\ln|1-y| + c$

$= x + 4\sqrt{x} + 4\ln|1-\sqrt{x}| + c$

$$\begin{array}{r|l} & y+2 \\ 1-y & y^2+y \\ & -y^2+y \\ \hline & 2y \\ & -2y+2 \\ \hline & 2 \end{array}$$

H.W.

1. $\int \frac{1}{1+\sqrt[4]{x}}\,dx$

let $\sqrt[4]{x} = y \Rightarrow y^4 = x \Rightarrow dx = 4y^3\,dy$

$\therefore \int \frac{1}{1+\sqrt[4]{x}}\,dx = \int \frac{1}{1+y}\,4y^3\,dy$

$= 4\int \frac{y^3}{1+y}\,dy$

$= 4\left[\int (y^2+y+1)\,dy - \int \frac{1}{1+y}\,dy\right]$

$$\begin{array}{r|l} & y^2+y+1 \\ 1+y & y^3 \\ & \mp y^3 \mp y^2 \\ \hline & -y^2 \\ & \mp y^2 + y \\ \hline & y \\ & \mp y \mp 1 \\ \hline & -1 \end{array}$$

$$= 4\frac{y^3}{3} + 4\frac{y^2}{2} + 4y - \ln|1+y| + c$$

$$= \frac{4}{3}(\sqrt[4]{x})^3 + 2(\sqrt[4]{x})^2 - 4(\sqrt[4]{x}) - \ln|1+\sqrt[4]{x}| + c$$

$$= \frac{4}{3}(x)^{\frac{3}{4}} + 2(x)^{\frac{1}{2}} - 4(x)^{\frac{1}{4}} - \ln|1+(x)^{\frac{1}{4}}| + c$$

2. $\int 2\sqrt{2x-1}\,dx$

let $y = \sqrt{2x-1} \Rightarrow y^2 = 2x-1 \Rightarrow x = \frac{y^2-1}{2}$

$$dx = \frac{2(2y)-(y^2-1)(0)}{4} = y\,dy$$

$$\therefore \int 2\sqrt{2x-1}\,dx = \int 2y \cdot y\,dy$$

$$= 2\int y^2\,dy$$

$$= 2\frac{y^3}{3} + c$$

$$= \frac{2}{3}(\sqrt{2x-1})^3 + c$$

$$= \frac{2}{3}(2x-1)^{\frac{3}{2}} + c$$

3. $\int x\sqrt{2x-1}\,dx$

let $y=\sqrt{2x-1} \Rightarrow y^2 = 2x-1 \Rightarrow x = \frac{y^2+1}{2}$

$dx = y\,dy$

$\therefore \int x\sqrt{2x-1}\,dx = \int \frac{y^2+1}{2} \cdot y\,y\,dy$

$= \int \frac{y^2+1}{2} y^2\,dy = \int \frac{y^4+y^2}{2}\,dy$

$= \frac{1}{2}\left(\frac{y^5}{5} + \frac{y^3}{3}\right) + c$

$= \frac{1}{2}\left(\frac{(\sqrt{2x-1})^5}{5} + \frac{(\sqrt{2x-1})^3}{3}\right) + c$

$= \frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + c$

4. $\int \sqrt{\frac{1+x}{1-x}}\,dx$

let $y = \sqrt{\frac{1+x}{1-x}} \Rightarrow y^2 = \frac{1+x}{1-x}$

$y^2 - y^2x = 1+x$

$y^2 - 1 = x + xy^2$

$y^2 - 1 = x(1+y^2)$

$x = \frac{y^2-1}{1+y^2} \Rightarrow dx = \frac{4y}{(y^2+1)^2}\,dy$

$$\therefore \int \sqrt{\frac{1+x}{1-x}}\,dx = \int y \frac{4y}{(y^2+1)^2}\,dy$$

$$\int \frac{4y^2}{(y^2+1)^2}dy = 4\left(\int \frac{Ay+B}{y^2+1}\,\cancel{dy}\,dy + \int \frac{Cy+D}{(y^2+1)^2}dy\right)$$

$$\left(\frac{y^2}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2}\right) \times (y^2+1)^2$$

$$y^2 = (Ay+B)(y^2+1) + Cy + D$$

$$y^2 = Ay^3 + Ay + By^2 + B + Cy + D$$

$$Ay^3 = 0 \quad 1 \Rightarrow A = 0$$
$$By^2 = y^2 \quad 2 \Rightarrow B = 1$$

$$Ay + Cy = 0 \quad 3 \Rightarrow A = -C \Rightarrow C = 0$$
$$B + D = 0 \quad 4 \Rightarrow B = -D \Rightarrow D = -1$$

$$\therefore \int \frac{4y^2}{(y^2+1)^2}dy = 4\left[\int \frac{1}{y^2+1}dy + \int \frac{-1}{(y^2+1)^2}dy\right]$$

$$= 4\left(\tan^{-1}y + \frac{1}{y^2+1}\right) + c$$

$$= 4\tan^{-1}y + \frac{4}{y^2+1} + c$$

$$= 4\tan^{-1}\sqrt{\frac{1+x}{1-x}} + \frac{4}{\frac{1+x}{1-x}+1} + c$$