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Open set, Closed set, Neighborhood

There is a considerable of concepts which is play a role in connection with (metric space, normed space) and Topological space, we start with concept open set.

Definition Open set.

Let $x_0 \in X$ (with X is normed (metric space) and real number $r > 0$. we define

① open ball

$$B_r(x_0) = \{x \in X, \text{ or } \begin{matrix} d(x, x_0) < r \\ \| x - x_0 \| < r \end{matrix} \}$$

نصف قطر ← r , المركز ← x_0

in metric ← $d(x, x_0) < r$

in normed space ← $\| x - x_0 \| < r$

② closed ~~set~~ ball

$$\overline{B_r}(x_0) = \{x \in X, \text{ or } \begin{matrix} d(x, x_0) \leq r \\ \| x - x_0 \| \leq r \end{matrix} \}$$

in metric ← $d(x, x_0) \leq r$

in normed ← $\| x - x_0 \| \leq r$

③ sphere

$$S_r(x_0) = \{x \in X, \text{ or } \begin{matrix} d(x, x_0) = r \\ \| x - x_0 \| = r \end{matrix} \}$$

x_0 is called the center ~~of the~~

r is called the radius

we see $S_r(x_0) = \overline{B_r}(x_0) - B_r(x_0)$

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Definition Open set

A subset M of a metric space X is said to be open if it contains a ball about each of its points. or it is a union of open ball.

A subset ~~W~~ W of X is said to be closed if its complement is open, that is $W^c = X - W$

note ① every metric space is topological space

② every normed space is topological space.

Definition Closure set

~~A subset~~ Let M be a subset of space X, then The closure of M, which is denoted by $\overline{M}$ is intersection of all closed set in X which is contain M.

$$\overline{M} = \{\cap H : M \subseteq H,\ H \text{ is closed}\}$$

Theorem Let M be a nonempty subset of a metric (normed) space X and $\overline{M}$ its closure

① $x \in \overline{M}$ if and only if ~~there~~ there is a sequence (x_n) in M such that $x_n \to x$

② M is closed if and only if $(x_n) \subseteq M$, $x_n \to x$ implies that $x \in M$.

Note

A subspace of Banach space need not to be Banach. The condition it make it Banach shown in the following proposition.

proposition Let M be a subspace of Banach space X. Then M is Banach space if and only if M is closed set.

proof: Let M be a Banach space and $(x_n)_{n=1}^{\infty}$ is a sequence in M which is converges to $x \in X$

$\because (x_n)_{n=1}^{\infty}$ is convergent sequence, so it is Cauchy sequence by using "Every convergent sequence in normed space is Cauchy sequence"

$\because M$ is Banach space so $(x_n)_{n=1}^{\infty}$ is convergent to point $x_0 \in M$, since by using Theorem "Every convergent sequence in normed space convergent to unique point on it", so $x = x_0$ and $x \in M$ by using Theorem

"M is closed if and only if $(x_n)_{n=1}^{\infty} \in M$, $x_n \to x$ implies $x \in M$"

$\therefore M$ is closed set

Conversely

Let M be a closed set and $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in M, $\because \bar{X}$ is a Banach space. Then $(x_n)_{n\geq 1}^{\infty}$ is a convergent sequence and convergent to $x_n \longrightarrow x$ and $x \in \bar{X}$

$\because M$ is closed set by using Theorem

" M is closed if and only if $(x_n)_{n\geq 1}^{\infty} \in M$, $x_n \longrightarrow x$ ~~The~~ implies $x \in M$ "

So $x \in M$

$\therefore$ M is a Banach space.

~~Product space~~

~~Let $\bar{X}$, $\bar{Y}$ be two vectors space over field k, the vectors.~~

Example R^3 over R is Banach space and $M_1 = \{ (x, 0, 0) / x \in R \}$

$M_2 = \{ (0, x, y) / x, y \in R \}$

M_1 and M_2 are two subspace of Banach space, M_1, M_2 are ~~closed~~ Banach space so M_1, M_2 are closed set.

Example Let The normed space $C[0,2]$ with norm. $\|f\| = \int_0^2 |f(t)|\,dt$, for $f \in C[0,2]$

Then $(C[0,2], \|\cdot\|)$ is not Banach space.

Solution consider the following sequence of functions

$$f_n(t) = \begin{cases} t^n & 0 \leq t \leq 1 \\ 1 & 1 < t \leq 2 \end{cases}$$

we to show that is a Cauchy sequence $(C[0,2], \|\cdot\|)$

Let $n < m$

$$\|f_n - f_m\| = \int_0^2 |f_n(t) - f_m(t)|\,dt$$

$$= \int_0^1 |f_n(t) - f_m(t)|\,dt + \int_1^2 |f_n(t) - f_m(t)|\,dt$$

$$= \int_0^1 |t^n - t^m|\,dt + \int_1^2 (1-1)\,dt$$

$\because n < m, \rightarrow t \in [0,1] \rightarrow t^n \geq t^m$

so $t^n - t^m > 0$

$$= \int_0^1 (t^n - t^m)\,dt$$

$$= \left. \frac{t^{n+1}}{n+1} - \frac{t^{m+1}}{m+1} \right]_0^1$$

$$= \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1} < \epsilon \quad \forall\, n,m > K$$

so $(f_n)_{n=1}^{\infty}$ is a Cauchy Sequence in $C[0,2], \|\cdot\|)$.

now, let us to show that $(f_n)_{n=1}^{\infty}$ is convergent to a non continuous function:

suppose that $f_n \longrightarrow f_0$, then

$$\|f_n - f_0\| = \int_0^1 |f_n(t) - f_0(t)|\,dt + \int_1^2 |f_n(t) - f_0(t)|\,dt$$

$$= \int_0^1 |t^n - f_0(t)|\,dt + \int_1^2 |1 - f_0(t)|\,dt$$

since

$$|f_0(t)| - |t^n| \le |t^n - f_0(t)| \le |f_0(t)| + |t^n|$$

implies that

$$\int_0^1 |f_0(t)|\,dt - \int_0^1 |t^n|\,dt \le \int_0^1 |t^n - f_0(t)|\,dt \le \int_0^1 |f_0(t)|\,dt + \int_0^1 |t^n|\,dt$$

$$\int_0^1 |t^n|\,dt = \int_0^1 t^n\,dt \qquad \text{since } t^n > 0 \text{ for } t \in [0,1]$$

$$= \frac{t^{n+1}}{n+1}\Big]_0^1 = \frac{1}{n+1} \longrightarrow 0$$

as $n \to \infty$

so when $n \to \infty$

$$\int_0^1 |f_0(t)|\,dt \le \cancel{|t^n - f_0(t)|} \lim_{n\to\infty} \int_0^1 |t^n - f_0(t)|\,dt \le \int_0^1 |f_0(t)|\,dt$$

so $\int_0^1 |t^n - f_0(t)|\,dt \longrightarrow \int_0^1 f_0(t)\,dt$

So

$$\|f_n - f_0\| = \int_0^1 |f_0(t)|\,dt + \int_1^2 |1 - f(t)|\,dt$$

when $n \to \infty$

Since the integral term is non-negative, so must the right hand side terms approaches to zero

$$f_0(t) = \begin{cases} 0 & 0 \leq t \leq 1 \\ 1 & 1 < t \leq 2 \end{cases}$$

we see that $f_0(t)$ not continuous function at $t = 1$, so $f_0 \notin C[0,2]$, that is mean f_n is convergent to point not belong to the same space; So $C[0,2]$ is not Banach space.

Remark Every subspace of normed space is also normed space but not necessarily every subspace of a Banach space is Banach as shown in the following example.

Example

Let $C[0,1]$ the space of continuous function over R with $\|f\| = \max\limits_{x \in [a,b]} |f(x)|$

so $(C[0,1], \|\cdot\|)$ is Banach space

consider. The subspace $P_n(X)$ the span of polynomial

$$P_n(x) = \{ \sum_{i=1}^{n} x^k / n \in \mathbb{N} \} \; x \in [0,1] \}$$

~~Solution~~

It is clearly that $P_n(X)$ is a sub space of $C[0,1]$, and it is normed space but $P_n(X)$ is not Banach space since the sequence $\left(\sum_{i=1}^{n} x^k \right)_{n=1}^{\infty}$

is a sequence in $P_n(X)$ and it is convergent to e^x but

$$e^x \notin P_n(x)$$

so $P_n(V)$ is a ~~subspace~~ normed space but not Banach space.