1.2 Orthogonal Complements

Definition (orthogonality)

An element x on a pre Hilbert space X is said to be orthogonal to an element $y \in X$ if $\langle x / y \rangle = 0$

we also say that x and y are orthogonal and we write $x \perp y$. Similarly for a subsets $A, B \subseteq X$ we write $x \perp A$ if $x \perp a$ for all $a \in A$ and we say $A \perp B$ if $a \perp b$ for all $a \in A$, $b \in B$

Example Let $\mathbb{C}^2$ be a vector space over $\mathbb{C}$

$x = (3+2i, -i)$ $\qquad$ $A = (\mathbb{C}, 0)$

$y = (-2i, 6-4i)$ $\qquad$ $B = (0, \mathbb{C})$

$z = (2i, 0)$

$w = (0, 3i)$

we have $x \perp y$, $z \perp w$, $z \perp B$, $w \perp A$, $A \perp B$

solution

$\langle x / y \rangle = (3+2i)\overline{(-2i)} + -i\,\overline{(6-4i)}$

$= (3+2i)(2i) + -i(6+4i)$

$= 6i - 4 + -6i + 4$

$= 0$

$\langle z, w\rangle = 2i\,\overline{(0)} + 0 \times \overline{3i} = 0$

Let $b \in B \rightarrow b = (0, b_2)$ $b_1 \in \mathbb{C}$

$\langle z, b\rangle = 2i\,\overline{(0)} + 0 \times \overline{b_2}$

$= 0$

$\langle z, b\rangle = 0 \quad \forall\, b \in B$

$\therefore z \perp B$

Let $a \in A$, $a = (a_1, 0)$ $a_1 \in \mathbb{C}$

$\langle w, a\rangle = 0 \times \overline{a_1} + 3i\,\overline{0}$

$= 0$

$\langle w, a\rangle = 0 \quad \forall\, a \in A$

$\therefore w \perp A$

$\forall\, a \in A, b \in B$ $\quad a = (a_1, 0)$ $\quad a_1, b_2 \in \mathbb{C}$
$\quad b = (0, b_2)$

$\langle a, b\rangle = a_1 \cdot \overline{0} + 0 \cdot \overline{b_2}$

$= 0 \quad \forall\, a \in A \;\&\; b \in B$

$\therefore A \perp B.$

Note Let X be a pre Hilbert space. $x, y \in X$

if $\langle x, y\rangle = 0$, so $\langle y, x\rangle = 0$

if $x \perp y$ implies $y \perp x$

$\langle x, y\rangle = 0 \quad \langle y, x\rangle = \overline{\langle x, y\rangle} = 0 \longrightarrow \overline{\langle y, x\rangle} = 0 \longrightarrow \overline{\overline{\langle y, x\rangle}} = \overline{0}$

Example in previous example $x = (3+2i, -i) \rightarrow \langle y, x\rangle = 0$
$y = (-2i, 6-4i)$

we see that $\langle x, y\rangle = 0$

now $\langle y, x\rangle = -2i\,\overline{(3+2i)} + (6-4i)\,\overline{(-i)}$

$= -2i(3-2i) + (6-4i)\, i$

$= -6i - 4 + 6i + 4$

$= 0$

Lemma Let $\overline{X}$ be a pre Hilbert space

$x \in \overline{X}$ and $y_i \in \overline{X}$ $i = 1, \ldots n$, $n \in N$

if $x \perp y_i$ $\forall i = 1, \ldots n$ then

for $z = \sum_{i=1}^{n} \alpha_i y_i$, $\alpha_i \in K$, ~~Tho~~ $x \perp z$

proof:

$\langle x / z \rangle = \langle x / \sum_{i=1}^{n} \alpha_i y_i \rangle$

$= \langle x / \alpha_1 y_1 \rangle + \langle x, \alpha_2 y_2 \rangle \cdots \langle x, \alpha_n y_n \rangle$

$= \overline{\alpha_1} \underbrace{\langle x / y_1 \rangle}_{=0} + \overline{\alpha_2} \underbrace{\langle x, y_2 \rangle}_{=0} \cdots \overline{\alpha_n} \underbrace{\langle x, y_n \rangle}_{=0}$

$= \overline{\alpha_1} \cdot 0 + \overline{\alpha_2} \cdot 0 \cdots \overline{\alpha_n} \cdot 0$

$= 0$

Example Let C^2 over C

$x = (2i, 0$

$y_1 = (0, 3i)$ $\alpha_1 = 2$

$y_2 = (0, 1+i)$ $\alpha_2 = i$

~~$y = \alpha_1 y_1 + \alpha_2 y_2$~~

$x \perp y_1$ $\langle x, y_1 \rangle = 2i\,\overline{0} + 0\,\overline{3i} = 0$

$x \perp y_2$ $\langle x, y_2 \rangle = 2i\,\overline{0} + 0\,\overline{(1+i)} = 0$

$y = \alpha_1 y_1 + \alpha_2 y_2$

$= 2(0, 3i) + i(0, 1+i)$

$= (0, 6i) + (0, i-1)$

$= (0, -1+7i)$

$\langle x, y \rangle = 2i\,\overline{(0)} + 0\,\overline{(-1+7i)} = 0$

<u>Lemma</u> let X be an pre Hilbert space over K

~~let~~ for $x, y \in X$. Then

$x \perp y$ if and only if $\|x+\alpha y\| = \|x-\alpha y\|$ $\forall \alpha \in K$

<u>proof</u> Let $x \perp y$ that is mean $\langle x/y\rangle = 0$

also $\langle y/x\rangle = \overline{\langle x/y\rangle} = \bar{0} = 0$

Then $\langle y/x\rangle = 0$

Let $\alpha \in K$

~~$\|x$~~ we take left hand side $\|x+\alpha y\|$

$$\|x+\alpha y\|^2 = \langle x+\alpha y / x+\alpha y\rangle$$

$$= \langle x/x\rangle + \alpha\overset{0}{\langle y, x\rangle} + \bar{\alpha}\overset{0}{\langle x, y\rangle} + \alpha\bar{\alpha}\langle y/y\rangle$$

$$= \|x\|^2 + \alpha \cdot 0 + \bar{\alpha}\cdot 0 + |\alpha|^2\|y\|^2$$

$$\because \alpha\bar{\alpha} = |\alpha|^2$$

$$\|x+\alpha y\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2$$

now we take the right hand side

$$\|x-\alpha y\|^2 = \langle x-\alpha y / x-\alpha y\rangle$$

$$= \langle x/x\rangle - \alpha\langle y/x\rangle - \bar{\alpha}\langle x, y\rangle + \alpha\bar{\alpha}\langle y/y\rangle$$

$$= \|x\|^2 - \alpha\cdot 0 - \bar{\alpha}\cdot 0 + |\alpha|^2\|y\|^2$$

$$\|x-\alpha y\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2$$

$$\therefore \|x+\alpha y\|^2 = \|x-\alpha y\|^2$$

we get $\|x+\alpha y\| = \pm\|x-\alpha y\|$ since the norm ~~cannot~~ can not be negtive

so $\|x+\alpha y\| = \|x-\alpha y\|$

<u>Conversely</u> Let $\|x+\alpha y\| = \|x-\alpha y\| \quad \forall \alpha \in K$

then $\|x+\alpha y\|^2 = \|x-\alpha y\|^2$ so

$$\langle x+\alpha y / x+\alpha y\rangle = \langle x-\alpha y / x-\alpha y\rangle$$

$$\cancel{\langle x/x\rangle} + \bar{\alpha}\langle x/y\rangle + \alpha\langle y/x\rangle + \alpha\bar{\alpha}\cancel{\langle y,y\rangle}$$

$$= \cancel{\langle x/x\rangle} - \bar{\alpha}\langle x/y\rangle - \alpha\langle y/x\rangle + \alpha\bar{\alpha}\cancel{\langle y,y\rangle}$$

$$\bar{\alpha}\langle x/y\rangle + \alpha\langle y/x\rangle = -\bar{\alpha}\langle x/y\rangle - \alpha\langle y/x\rangle$$

$$\bar{\alpha}\langle x/y\rangle + \bar{\alpha}\langle x/y\rangle + \alpha\langle y/x\rangle + \alpha\langle y/x\rangle = 0$$

$$\frac{2}{2}\bar{\alpha}\langle x/y\rangle + \frac{2\alpha\langle y/x\rangle}{2} = \frac{0}{2}$$

so $\bar{\alpha}\langle x/y\rangle + \alpha\langle y/x\rangle = 0$ $(*)$ this is true $\forall \alpha \in K$

if X is vector space over R, then

$\alpha = \bar{\alpha}$ and $\langle x/y\rangle = \langle y/x\rangle$

so $\alpha\langle x/y\rangle + \alpha\langle x/y\rangle = 0$

$2\alpha\langle x/y\rangle = 0 \rightarrow \langle x/y\rangle = 0$

if X is a vector space over $\mathbb{C}$, then take $\alpha = 1$ ~~ones then~~ at first, then take $\alpha = i$ in equation $(*)$

$$1\langle x/y\rangle + 1\langle y/x\rangle = 0$$

$$\bar{i}\langle x/y\rangle + i\langle y/x\rangle = 0$$

$\langle x|y\rangle + \langle y|x\rangle = 0$ $\quad x^{-}i$

$-i\langle x|y\rangle + i\langle y|x\rangle = 0$

$-i\langle x|y\rangle + -i\langle y|x\rangle = 0$

$-i\langle x|y\rangle + i\langle y|x\rangle = 0$ by adding

$\cancel{-2i}\langle x|y\rangle = 0$

$\overline{\cancel{-2i}} \qquad \overline{-2i}$

$\langle x|y\rangle = 0$

so $x \perp y$

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Example

~~...~~ In $\mathbb{C}^2$

for $z \in \mathbb{C}^2$, $z = (z_1, z_2)$

$\|z\| = \sqrt{|z_1|^2 + |z_2|^2}$

$x = (3+2i, -i)$

$y = (-2i, 6-4i)$

we have $x \perp y$ is $\|x+\alpha y\| = \|x-\alpha y\|$

Solution

$x+\alpha y = (3+2i, -i) + \alpha(-2i, 6-4i)$

$= (3+2i, -i) + (-2\alpha i, 6\alpha - 4\alpha i)$

$= (3+2i-2\alpha i, -i+6\alpha-4\alpha i)$

$= (3+2(1-\alpha)i, 6\alpha - (1+4\alpha)i)$

$\|x+\alpha y\| = \sqrt{|3+2(1-\alpha)i|^2 + |6\alpha-(1+4\alpha)i|^2}$

$= \sqrt{9+4(1-\alpha)^2 + 36\alpha^2 + (1+4\alpha)^2}$

$= \sqrt{9+4\cancel{-8\alpha}+4\alpha^2+36\alpha^2+1+\cancel{8\alpha}+16\alpha^2}$

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$= \sqrt{14+56\alpha^2}$

$$x-\alpha y = (3+2i, -i) - \alpha(-2i, 6-4i)$$
$$= (3+2i, -i) + (2\alpha i, -6\alpha + 4\alpha i)$$
$$= (3+2i+2\alpha i, -i-6\alpha+4\alpha i)$$
$$= (3+2(1+\alpha)i, -6\alpha - (1-4\alpha)i)$$

$$\|x+\alpha y\| = \sqrt{|3+2(1+\alpha)i|^2 + |-6\alpha-(1-4\alpha)i|^2}$$
$$= \sqrt{9+4(1+\alpha)^2+36\alpha^2+(1-4\alpha)^2}$$
$$= \sqrt{9+4(1+2\alpha+\alpha^2)+36\alpha^2+1-8\alpha+16\alpha^2}$$
$$= \sqrt{9+4+\cancel{8\alpha}+4\alpha^2+36\alpha^2+1-\cancel{8\alpha}+16\alpha^2}$$
$$= \sqrt{14+56\alpha^2}$$

$\therefore \|x+\alpha y\| = \|x-\alpha y\|$.

Pythagorean Theorem

In pre-Hilbertspace X, for $x, y \in X$
if $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

proof: since $x \perp y$, so $\langle x/y \rangle = 0$

$$\|x+y\|^2 = \langle x+y / x+y \rangle$$
$$= \langle x/x \rangle + \langle x/y \rangle + \langle y/x \rangle + \langle y/y \rangle$$
$$= \|x\|^2 + 0 + 0 + \|y\|^2$$

so $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

<u>Example</u> $x = (3+2i, -i)$

$y = (-2i, 6-4i)$

It is clearly that $x \perp y$ so

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$$\# \; x+y = (3+2i, -i) + (-2i, 6-4i)$$

$$= (3 + \cancel{2i} - \cancel{2i}, -i + 6 - 4i)$$

$$= (3, 6-5i)$$

$$\|x+y\| = \sqrt{3^2 + 6^2 + (-5)^2} =$$

$$= \sqrt{9 + 36 + 25}$$

$$= \sqrt{70}$$

$$\|x+y\|^2 = 70$$

$$\underline{\|x\|^2} = |3+2i|^2 + |-i|^2$$

$$= 9 + 4 + 1 = 14$$

$$\|y\|^2 = |-2i|^2 + |6-4i|^2$$

$$= 4 + 36 + 16 = 56$$

$$\|x+y\|^2 = 70 = 14 + 56 = \|x\|^2 + \|y\|^2$$

Existence and uniqueness problem

مسألة الوجود والوحدانية

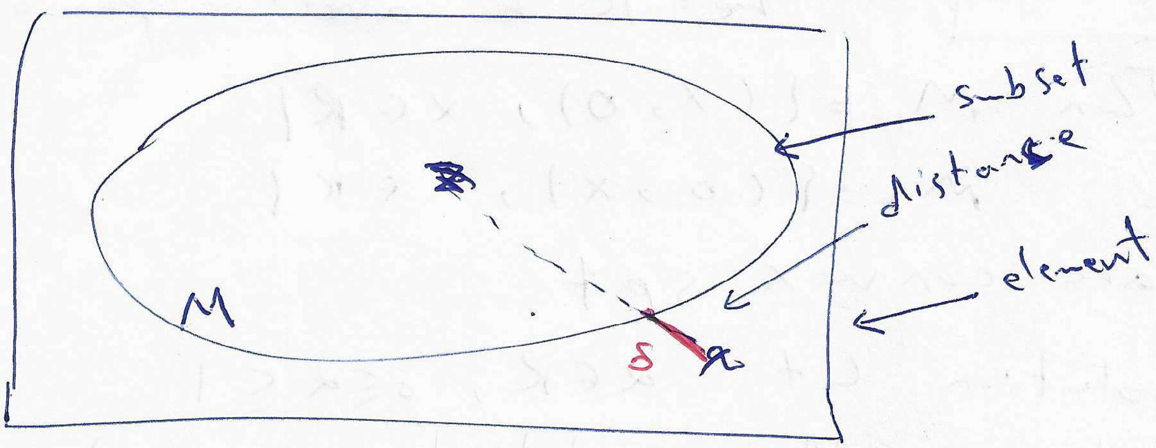
In a metric space X the distance δ from an element x to a nonempty subset $M \subseteq X$ is defined to be

$$\delta = \inf_{z \in M} \{d(x, z)\}$$

where $(M \neq \phi)$

In a normed space this becomes

$$\delta = \inf_{z \in M} \{\|x - z\|\}$$



X

The existence problem is there a element $y \in M$ such that $\delta = \|x - y\|$ —— (1)

and the uniqueness is y is the unique element that satisfies (1)

This problem is fundamental importance theoretically as well as in applications, for instance in connection with approximations of functions

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The study of this problem of Hilbert spaces is simpler than that of general Banach spaces so we start with following definition

Definition Convex Set مجموعة محدبة

A subset M of vector space X is said to be convex, if for every $x, y \in \cancel{X} M$

$\alpha x + (1-\alpha) y \in M \quad (\alpha \in R, 0 \leq \alpha \leq 1)$

in another term

$\alpha M + (1-\alpha) M \subseteq M \quad \{\forall \alpha \in R, 0 \leq \alpha \leq 1\}$

Example Let R^2 be a vector space over R

Then $M_1 = \{(x, 0), x \in R\}$

$M_2 = \{(0, x), x \in R\}$

are convex set

solution Let $\alpha \in R, 0 \leq \alpha \leq 1$

Let $x, y \in M_1$, such that $x = (x_1, 0)$

$y = (y_1, 0)$

$\alpha(x_1, 0) + (1-\alpha)(y_1, 0) = (\alpha x_1, 0) + ((1-\alpha) y_1, 0)$

$= (\alpha x_1 + (1-\alpha) y_1, 0)$

$\because \alpha x_1 + (1-\alpha) y_1 \in R$, so

$(\alpha x_1 + (1-\alpha) y_1, 0) \in M_1$

$\therefore M_1$ is convex

Similarly for M_2 set