Line Integral
التكامل الخطي

Theorem:- Let f be continuous in region containing a smooth curve (منحني املس) c. If c is given by $r(t) = x(t)i + y(t)j$ where $a \le t \le b$, then

$$\int_c f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{[\acute{x}(t)]^2 + [\acute{y}(t)]^2}\, dt$$

where $ds = \sqrt{[\acute{x}(t)]^2 + [\acute{y}(t)]^2}\, dt$

If C is given by $r(t) = x(t)i + y(t)j + z(t)k$ **,** where $a \le t \le b$, then

$$\int_c f(x,y,z)\, ds = \int_a^b f\big(x(t), y(t), z(t)\big) \sqrt{[\acute{x}(t)]^2 + [\acute{y}(t)]^2 + [\acute{z}(t)]^2}\, dt$$

Example 1: Evaluate $\int_c (x^2 - y + 3z)\, ds$, where c is represented by the parametric equations $x = t,\ y = 2t,\ z = t$, where $0 \le t \le 1$

Solution:

$$\acute{x}(t) = 1, \ \acute{y}(t) = 2, \quad \acute{z}(t) = 1$$

$$\int_c (x^2 - y + 3z)\, ds = \int_0^1 (t^2 - 2t + 3t) \sqrt{(1)^2 + (2)^2 + (1)^2}\, dt$$

$$\int_0^1 (t^2 + t)\sqrt{6}\, dt = \sqrt{6} \int_0^1 (t^2 + t)\, dt = \sqrt{6}\ \frac{t^3}{3} + \frac{t^2}{2}\Big]_0^1$$

$$= \sqrt{6}\ \left[\frac{1}{3} + \frac{1}{2} - 0\right] = \sqrt{6}\left(\frac{2+3}{6}\right) = \sqrt{6}\left(\frac{5}{6}\right) = \frac{5}{\sqrt{6}}$$

Example 2: Evaluate $\int_c (1 + xy^2)\, ds$, from (0,0) to (1,2) along the line segment c that is represented by the parametric equations $x = 3t,\ y = 2t,$, where $0 \le t \le 4$

Solution:

$$\acute{x}(t) = 3, \ \acute{y}(t) = 2,$$

$$\int_c (1 + xy^2)\, ds = \int_0^4 [1 + (3t)(2t)^2] \sqrt{(3)^2 + (2)^2}\, dt$$

$$= \sqrt{13} \int_0^4 (1 + 12t^3)\, dt = \sqrt{13}\ (t + 12\frac{t^4}{4}\Big]_0^4 = \sqrt{13}(4 + 3(4)^4) =$$

Example 3: Evaluate $\int_c (xy - z^3)\, ds$, from $(1,0,0)$ to $(-1,0,\pi)$ along the line segment c that is represented by the parametric equations $x = cost,\ y = sin\, t, z = t$, where $0 \le t \le \pi$

Solution:

$$\acute{x}(t) = -sint,\ \ \acute{y}(t) = cost, \qquad \acute{z}(t) = 1$$

$$\int_c (xy - z^3)\, ds = \int_0^{\pi} [cost\ sin\, t - t^3] \sqrt{(-sint)^2 + (cost)^2 + (1)^2}\, dt$$

$$= \sqrt{2} \int_0^{\pi} (cost \sin t - t^3)\ dt = \sqrt{2}\ \frac{sin^2 t}{2} - \frac{t^4}{4}\Big]_0^{\pi} = \sqrt{2}\left(\frac{0}{2} - \frac{\pi^4}{4}\right) - 0$$

$$= -\frac{\sqrt{2}}{4}\pi^4$$

Example 4: Evaluate line integral $\int_c 2y\, dx + x^2\, dy$, along the curve $y = x^2$ from $(-2,4)$ to $(1,1)$

Solution:

$$y = x^2,\ dy = 2x\, dx$$

$$\int_{-2}^{1} 2x^2\, dx + x^2\, (2x\, dx) = \int_{-2}^{1} 2x^2\, dx + 2x^3\, dx$$

$$= \int_{-2}^{1} (2x^2 + 2x^3) dx = 2\frac{x^3}{3} + \frac{x^4}{2}\Big]_{-2}^{1} = \frac{2}{3} + \frac{1}{2} - \left(\frac{-16}{3} + 8\right) = -\frac{3}{2}$$

Example 5: Evaluate line integral $\int_c (x + 2)\, ds$, where c is represented by $r(t) = ti + \frac{4}{3} t^{\frac{3}{2}} j + \frac{1}{2} t^2 k$, where $0 \le t \le 2$

Solution:

$$\acute{x}(t) = 1,\ \ \acute{y}(t) = 2t^{\frac{1}{2}}, \qquad \acute{z}(t) = t$$

$$\int_c (x + 2)\, ds = \int_0^2 (t + 2) \sqrt{(1)^2 + \left(2t^{\frac{1}{2}}\right)^2 + (t)^2}\, dt$$

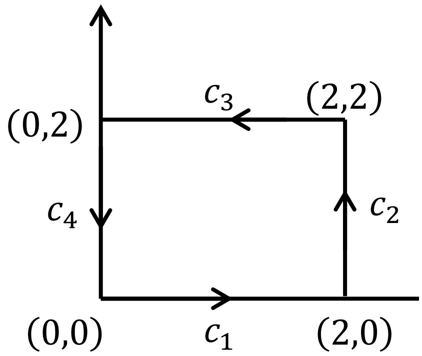
$$= \int_0^2 (t + 2) \sqrt{1 + 4t + t^2}\, dt = \frac{1}{2} \frac{(1 + 4t + t^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^2$$

If $c = c_1 + c_2 + \cdots + c_n$, such a curve is called piecewise smooth, we defined a line integral along a pricewise smooth curve c to be the sum of the integrals along section:

$$\int_c f(x,y,z)\, ds = \int_{c_1} f(x,y,z)\, ds + \int_{c_2} f(x,y,z)\, ds + \ldots + \int_{c_n} f(x,y,z)\, ds$$

Example 6: Evaluate $\int_c (x^3 - x^3 y)\, dx + (y^2 - 2xy)\, dy$ in a counter clockwise direction (باتجاه عكس عقارب الساعة) around the square path from $A(0,0), B(2,0), C(2,2), D(0,2)$.

Solution:



c_1: $y = 0 \Rightarrow dy = 0$
c_2: $x = 2 \Rightarrow dx = 0$
c_3: $y = 2 \Rightarrow dy = 0$
c_4: $x = 0 \Rightarrow dx = 0$

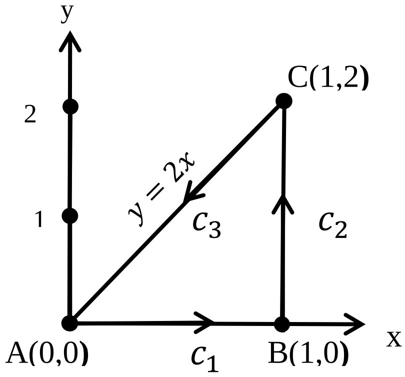
$$\int_c (x^3 - x^3 y)dx + (y^2 - 2xy)dy =$$

$$\int_0^2 (x^3 - x^3(0))\, dx + ((0) - 2x(0))0$$
$$+ \int_0^2 (8 - 8y)\,(0) + (y^2 - 4y)dy$$
$$+ \int_2^0 (x^3 - 2x^3)dx + (4 - 4x)0$$
$$+ \int_2^0 ((0) - (0)y)(0) + (y^2 - 2(0)y)dy$$

$$= \int_0^2 x^3\, dx + \int_0^2 (y^2 - 4y)dy + \int_2^0 -x^3 dx + \int_2^0 y^2 dy$$
$$= \left.\frac{x^4}{4}\right]_0^2 + \left.\frac{y^3}{3} - 2y^2\right]_0^2 - \left.\frac{x^4}{4}\right]_2^0 + \left.\frac{y^3}{3}\right]_2^0$$

$$= 4 + \frac{8}{3} - 8 - (\mathbf{0} - \mathbf{4}) + \left(\mathbf{0} - \frac{8}{3}\right) = 0$$

Example 7: Evaluate $\int_c x^2ydx + x\,dy$ in a counterclockwise direction (باتجاه عكس عقارب الساعة) around the triangular path from $A(0,0), B(1,0), C(1,2)$.

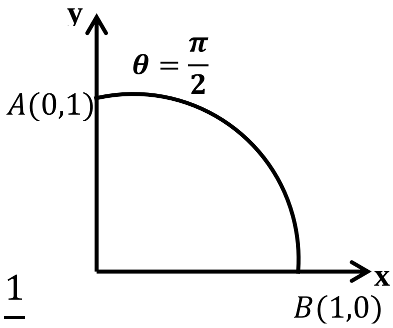


Solution:
c_1: $y = 0 \quad \Rightarrow \quad dy = 0$
c_2: $x = 1 \quad \Rightarrow \quad dx = 0$
c_3: $y = 2x, dy = 2\,dx$

Note: $\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} \quad \Longrightarrow \quad \frac{x-1}{0-1} = \frac{y-2}{0-2} \Longrightarrow \quad y = 2x$
$dy = 2\,dx$

$$= \int_0^1 0\,dx + \int_0^2 dy + \int_1^0 (2x^3 + 2x)dx = 0 + y]_0^2 + \frac{x^4}{2} + x^2\Big]_1^0$$
$$= 0 + 2 + \mathbf{0} - \left(\frac{1}{2} + 1\right) = 2 - \frac{3}{2} = \frac{1}{2}$$

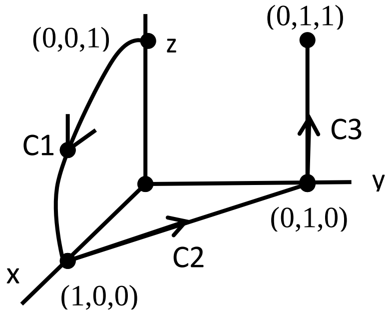
Example 8: Evaluate line integral $\int_c x^2\,ydy$ which lies(يقع) in the first quadrant arc (القوس) of the circle $x^2 + y^2 = 1$ in the xy-plane and the direction from $A(0,1)$ to $B(1,0)$. With the parametric representation $x = cos\theta,\ y = sin\,\theta$.



Solution:

$$\int_c x^2\,ydy = \int_{\frac{\pi}{2}}^0 cos^2\theta \sin\theta\,(cos\theta\,d\theta) =$$
$$\int_{\frac{\pi}{2}}^0 cos^3\theta\ sin\,\theta\,d\theta = -\frac{cos^4\theta}{4}\Big]_{\frac{\pi}{2}}^0 = -\frac{1}{4}[1-0] = -\frac{1}{4}$$

Example 9: Compute the value of $\int_c \ xz\,dx + x\,dy - yz\,dz$ along the oriented curve shown in the following figure which consisting of a quarter (نصف) circle in the xz- plane, and line segments(قطع) in the xy-plane and yz-plane, respectively(على التوالي) .



Solution:

$c1: x^2 + z^2 = 1 \Rightarrow z = \sqrt{1-x^2}, \ \ y = 0$

$c2: \frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} \ \ \Rightarrow \ \ \frac{x-1}{0-1} = \frac{y-0}{1-0} \Rightarrow \ \mathrm{x} = 1 - \mathrm{y}, z = 0$

$c3: \mathrm{y} = 1, x = 0$

$$\int_c \ xz\,dx + x\,dy - yz\,dz =$$

$$= \int_{c1} xz\,dx + x\,dy - yz\,dz + \int_{c2} xz\,dx + x\,dy - yz\,dz + \int_{c3} xz\,dx + x\,dy - yz\,dz$$

$$= \int_0^1 x\sqrt{1-x^2}\,dx \ + \int_0^1 (1-\mathrm{y})\,dy + \int_0^1 -z\,dz$$

$$= -\frac{1}{2}\frac{(1-x^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^1 + y - \frac{y^2}{2}\Bigg]_0^1 - \frac{z^2}{2}\Bigg]_0^1 = -\frac{1}{3}[0-1] + 1 - \frac{1}{2} - \frac{1}{2} = \frac{1}{3}$$

Example 10: Evaluate line integral $\int_c \ y^2\,dx - x\,dy$, along the curve $y^2 = 4x$ from (0,0) to (1,2).

Solution: With respect to x: $y^2 = 4x \ \ \Rightarrow \ \ y = 2\sqrt{x}$, $dy = x^{-\frac{1}{2}}\,dx$

$$\int_0^1 4x\,dx - x\,(x^{-\frac{1}{2}}\,dx) = \int_0^1 4x\,dx - x^{\frac{1}{2}}\,dx = \int_0^1 (4x - x^{\frac{1}{2}})\,dx$$

$$= 2x^2 - \frac{2}{3}x^{\frac{3}{2}}\Bigg]_0^1 = 2 - \frac{2}{3} = \frac{6-2}{3} = \frac{4}{3}$$

With respect to y : $y^2 = 4x \ \ \Rightarrow \ \ x = \frac{y^2}{4}$, $dx = \frac{y}{2}\,dy$

$$\int_0^2 y^2\,(\frac{y}{2}\,dy) - (\frac{y^2}{4})dy = \int_0^2 \frac{y^3}{2}dy - \frac{y^2}{4}\,dy = \int_0^2 (\frac{y^3}{2} - \frac{y^2}{4})\,dy$$

$$= \frac{y^4}{8} - \frac{y^3}{12}\Bigg]_0^2 = \frac{16}{8} - \frac{8}{12} = 2 - \frac{2}{3} = \frac{4}{3}$$

Example 11: Evaluate line integral $\int_c -y\,dx + x\,dy$, along the curve $y^2 = 4x$ from (4,4) to (0,0).

Solution: With respect to x: $y^2 = 4x \;\Rightarrow\; y = 2\sqrt{x}\;,\; dy = x^{-\frac{1}{2}}\,dx$

$$\int_4^0 -2\sqrt{x}\,dx + x\,(x^{-\frac{1}{2}}\,dx) = \int_4^0 -2x^{\frac{1}{2}}\,dx + x^{\frac{1}{2}}\,dx = \int_4^0 -x^{\frac{1}{2}}\,dx$$

$$= -\frac{2}{3}x^{\frac{3}{2}}\Big]_4^0 = -\left[0 - \left(\frac{16}{3}\right)\right] = \frac{16}{3}$$

With respect to y: $y^2 = 4x \;\Rightarrow\; x = \frac{y^2}{4}\;,\quad dx = \frac{y}{2}\,dy$

$$\int_4^0 -y\,(\frac{y}{2}\,dy) + (\frac{y^2}{4})\,dy = \int_4^0 -\frac{y^2}{2}dy + \frac{y^2}{4}\,dy = \int_4^0 -\frac{y^2}{4}\,dy$$

$$= -\frac{y^3}{12}\Big]_4^0 = -\left[0 - \frac{64}{12}\right] = \frac{16}{3}\,.$$