TEST YOUR KNOWLEDGE

1. Using $L(t) = \dfrac{1}{s^2}$, find the values of $L(t^2)$ and $L(t^3)$.

2. Using $L(\cos at) = \dfrac{s}{s^2 + a^2}$, find L(3 sin 8$t$).

3. Using $L(1) = \dfrac{1}{s}$ and $L(\sinh at) = \dfrac{a}{s^2 - a^2}$, find L(4 cosh 6$t$).

4. Using $L(\cosh at) = \dfrac{s}{s^2 - a^2}$, find L(2 sinh pt).

5. Using $L(1) = \dfrac{1}{s}$, find the Laplace transform of the function $\dfrac{1}{5} \sin \pi t$.

6. Using $L(1) = \dfrac{1}{s}$, find the Laplace transform of the function b cosh pt.

7. Using $L(1) = \dfrac{1}{s}$, find the Laplace transform of the function b sinh at.

8. Evaluate $L\left(\int_0^t e^{-x} \cos x \, dx\right)$.

Answers

1. $\dfrac{2}{s^3}, \dfrac{6}{s^4}$ **2.** $\dfrac{24}{s^2 + 64}$ **3.** $\dfrac{4s}{s^2 - 36}$ **4.** $\dfrac{2p}{s^2 - p^2}$

5. $\dfrac{\pi}{5(s^2 + \pi^2)}$ **6.** $\dfrac{bs}{s^2 - p^2}$ **7.** $\dfrac{ab}{s^2 - a^2}$ **8.** $\dfrac{s+1}{s(s^2 + 2s + 2)}$.

1.14. DIFFERENTIATION OF LAPLACE TRANSFORMS

Theorem. *Let f(t) be a real function such that*

(*i*) *f(t) is piecewise continuous on every finite interval in the range $t \geq 0$*

(*ii*) *there exists constants k and M such that $|f(t)| \leq Me^{kt}$ for $t \geq 0$, then*

$$F'(s) = -L(t\,f(t)), \text{ where } F(s) = L(f(t)).$$

Note. The proof of this theorem is beyond the scope of this book.

Remark. The result of the above theorem can be easily remembered as follows :

If L(f(t)) = F(s), then L(t f(t)) = – F′(s).

Corollary. We have $L(t\,f(t)) = -F'(s)$.

Applying this result successively, we get

$$L(t^2 f(t)) = (-1)^2 F''(s)$$

$$L(t^3 f(t)) = (-1)^3 F'''(s)$$

$$\cdots\cdots\cdots\cdots$$

$$\therefore \quad \mathbf{L(t^n f(t)) = (-1)^n F^{(n)}(s).}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Laplace transform of the following functions of t for $t \geq 0$:*

(i) $t\, e^{at}$ *(ii)* $t^2 \cosh \pi t$ *(iii)* $t^2 \sin 3t$.

Sol. *(i)* We have $L(e^{at}) = \dfrac{1}{s-a}$

$\therefore$ By **differentiability of Laplace transform,**

$$L(t\,.\,e^{at}) = -\frac{d}{ds}\left(\frac{1}{s-a}\right) = -\left(-(s-a)^{-2}\right) = \mathbf{\frac{1}{(s-a)^2}}.$$

(ii) We have $L(\cosh \pi t) = \dfrac{s}{s^2-\pi^2}$

$\therefore$ By **differentiability of Laplace transform,**

$$L(t\,.\cosh \pi t) = -\frac{d}{ds}\left(\frac{s}{s^2-\pi^2}\right) = -\left[\frac{(s^2-\pi^2)\,.\,1 - s\,.\,2s}{(s^2-\pi^2)^2}\right] = \frac{s^2+\pi^2}{(s^2-\pi^2)^2}$$

Again by using the same formula,

$$L(t\,.\,t\cosh \pi t) = -\frac{d}{ds}\left(\frac{s^2+\pi^2}{(s^2-\pi^2)^2}\right)$$

$$= -\left[\frac{(s^2-\pi^2)^2\,.\,2s - (s^2+\pi^2)\,.\,2(s^2-\pi^2)\,.\,2s}{(s^2-\pi^2)^4}\right] = \frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}.$$

$$\therefore \qquad L(t^2\cosh \pi t) = \mathbf{\frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}}.$$

Alternative method

$$L(t^2\cosh \pi t) = L\left(t^2\,.\,\frac{e^{\pi t}+e^{-\pi t}}{2}\right) = \frac{1}{2}\,L(e^{\pi t}\,t^2 + e^{-\pi t}\,t^2)$$

$$= \frac{1}{2}\,[L(e^{\pi t}\,t^2) + L(e^{-\pi t}\,t^2)] = \frac{1}{2}\left[\frac{2\,!}{(s-\pi)^3} + \frac{2\,!}{(s-(-\pi))^3}\right]$$

(Using $L(t^2) = \dfrac{2\,!}{s^3}$ and first shifting theorem)

$$= \frac{1}{(s-\pi)^3} + \frac{1}{(s+\pi)^3} = \frac{(s+\pi)^3+(s-\pi)^3}{(s^2-\pi^2)^3}$$

$$= \frac{2s^3+6s\pi^2}{(s^2-\pi^2)^3} = \frac{2s(s^2+3\pi^2)}{(s^2-\pi^2)^3}.$$

(iii) We have $L(\sin 3t) = \dfrac{3}{s^2+(3)^2} = \dfrac{3}{s^2+9}$

By **differentiability of Laplace transform,**

$$L(t\cdot\sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -3\left(-(s^2+9)^{-2}\cdot 2s\right) = \frac{6s}{(s^2+9)^2}$$

Again by using the same formula,

$$L(t\ .\ t \sin 3t) = -\frac{d}{ds}\left(\frac{6s}{(s^2+9)^2}\right)$$

$$= -6\left[\frac{(s^2+9)^2\ .\ 1 - s\ .\ 2\,(s^2+9)\ .\ 2s}{(s^2+9)^4}\right] = \frac{18(s^2-3)}{(s^2+9)^3}$$

$\therefore$ $$L(t^2 \sin 3t) = \mathbf{\frac{18(s^2-3)}{(s^2+9)^3}}.$$

Example 2. *Find the Laplace transform of the function $3te^{5t} \sin 7t$ for $t \geq 0$.*

Sol. We have $L(\sin 7t) = \dfrac{7}{s^2+49}$.

By **first shifting theorem,**

$$L(e^{5t} \sin 7t) = \frac{7}{(s-5)^2+49} = \frac{7}{s^2-10s+74} .$$

By **differentiability of Laplace transform,**

$$L(te^{5t} \sin 7t) = -\frac{d}{ds}\left(\frac{7}{s^2-10s+74}\right) = \frac{7\,.(2s-10)}{(s^2-10s+74)^2} = \frac{14\,(s-5)}{(s^2-10s+74)^2}$$

By **linearity of Laplace transform,**

$$L(3te^{5t} \sin 7t) = 3L(te^{5t} \sin 7t) = \mathbf{\frac{42\,(s-5)}{(s^2-10s+74)^2}} .$$

Example 3. *Prove that* $L\left[\dfrac{1}{2a^3}(\sin at - at \cos at)\right] = \dfrac{1}{(s^2+a^2)^2}$.

Sol. We have $L(\sin at) = \dfrac{a}{s^2+a^2}$. ...(1)

Also $L(\cos at) = \dfrac{s}{s^2+a^2}$

By **differentiability of Laplace transform,**

$$L\,(t \cos at) = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right).$$

$$= -\left[\frac{(s^2+a^2)\,.\,1 - s\,.\,2s}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

Now $L\left[\dfrac{1}{2a^3}(\sin at - at \cos at)\right]$

$$= \frac{1}{2a^3}\ [L(\sin at) - aL(t \cos at)]$$

$$= \frac{1}{2a^3}\left[\frac{a}{s^2+a^2} - a\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right)\right]$$

$$= \frac{1}{2a^3}\left[\frac{as^2+a^3-as^2+a^3}{(s^2+a^2)^2}\right] = \mathbf{\frac{1}{(s^2+a^2)^2}}.$$

Example 4. *Evaluate* $\int_0^\infty te^{-2t} \sin t \, dt$.

Sol. We have $\qquad L(\sin t) = \dfrac{1}{s^2+1^2} = \dfrac{1}{s^2+1}$.

$\therefore \qquad L(t \sin t) = -\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right) = -(-(s^2+1)^2 \,.\, 2s) = \dfrac{2s}{(s^2+1)^2}$

$\therefore \qquad \int_0^\infty e^{-st} \,.\, t \sin t \, dt = \dfrac{2s}{(s^2+1)^2}$

$\therefore \qquad \int_0^\infty e^{-2t} \, t \sin t \, dt = \left.\dfrac{2s}{(s^2+1)^2}\right|_{s=2} = \dfrac{2(2)}{((2)^2+1)^2} = \mathbf{\dfrac{4}{25}}$.

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L(f(t)) = F(s)$, then $L(t\, f(t)) = -F'(s)$.

Rule II. If $L(f(t)) = F(s)$, then

$L(t^2 f(t)) = (-1)^2 F''(s)$

$L(t^3 f(t)) = (-1)^3 F'''(s)$

....

$L(t^n f(t)) = (-1)^n F^{(n)}(s)$.

TEST YOUR KNOWLEDGE

1. Find the Laplace transform of the following functions of t for $t \ge 0$:

(*i*) $t\, e^{5t}$ (*ii*) $4t \cos 3t$ (*iii*) $6t \sin 5t$

(*iv*) $2t \sinh 2t$ (*v*) $3t \cosh 4t$.

2. Using $L(t^2) = \dfrac{2}{s^3}$, find the Laplace transform of t^3 and t^4.

3. Find the Laplace transform of the following functions of t for $t \ge 0$:

(*i*) $2t^2 e^{3t}$ (*ii*) $t^2 \cos 4t$ (*iii*) $t^2 \sin at$

(*iv*) $6t^2 \sinh 3t$ (*v*) $5t^2 \cosh 7t$.

4. Find the Laplace transform of the following functions of t for $t \ge 0$:

(*i*) $t^3 e^{2t}$ (*ii*) $3t^3 \cos \pi t$ (*iii*) $4t^3 \sin 4t$

(*iv*) $5t^3 \sinh 4t$ (*v*) $6t^3 \cosh (t/2)$.

5. Find the Laplace transform of the following functions of t for $t \ge 0$:

(*i*) $t\, e^{2t} \cos 4t$ (*ii*) $t\, e^{-t} \sin 3t$

(*iii*) $t\, e^{3t} \sinh 3t$ (*iv*) $3t\, e^{2t} \cosh 7t$ (*v*) $te^{-2t} \sin 3t$

6. Prove that $L\left(\dfrac{1}{2a}(\sin at + at \cos at)\right) = \dfrac{s^2}{(s^2+a^2)^2}$.

7. Evaluate $\int_0^\infty t^3 e^{-t} \sin t \, dt$.

Answers

1. (*i*) $\frac{1}{(s-5)^2}$ (*ii*) $\frac{4(s^2-9)}{(s^2+9)^2}$ (*iii*) $\frac{60s}{(s^2+25)^2}$

(*iv*) $\frac{8s}{(s^2-4)^2}$ (*v*) $\frac{3(s^2+16)}{(s^2-16)^2}$ **2.** $\frac{6}{s^4}, \frac{24}{s^5}$

3. (*i*) $\frac{4}{(s-3)^3}$ (*ii*) $\frac{2s(s^2-48)}{(s^2+16)^3}$ (*iii*) $\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

(*iv*) $\frac{108(s^2+3)}{(s^2-9)^3}$ (*v*) $\frac{10s(s^2+147)}{(s^2-49)^3}$ **4.** (*i*) $\frac{6}{(s-2)^4}$

(*ii*) $\frac{18(\pi^4-6\pi^2s^2-s^4)}{(s^2+\pi^2)^4}$ (*iii*) $\frac{384s(s^2-16)}{(s^2+16)^4}$ (*iv*) $\frac{480s(s^2+16)}{(s^2-16)^4}$

(*v*) $\frac{576(16s^4+24s^2+1)}{(4s^2-1)^4}$ **5.** (*i*) $\frac{s^2-4s-12}{(s^2-4s+20)^2}$ (*ii*) $\frac{6(s+1)}{(s^2+2s+10)^2}$

(*iii*) $\frac{6(s-3)}{(s^2-6s)^2}$ (*iv*) $\frac{3(s^2-4s+53)}{(s^2-4s-45)^2}$ (*v*) $\frac{6(s+2)}{(s^2+4s+13)^2}$

7. 0.

1.15. INTEGRATION OF LAPLACE TRANSFORMS

Theorem. *Let f(t) be a real function such that*

(*i*) *f(t) is piecewise continuous on every finite interval in the range $t \geq 0$*

(*ii*) *there exists constants k and M such that $|f(t)| \leq Me^{kt}$ for $t \geq 0$*

(*iii*) $\lim_{t \to 0^+} \frac{f(t)}{t}$ *exists finitely, then*

$$\int_s^\infty F(S)\, dS = L\left(\frac{f(t)}{t}\right), s > k \text{ where } F(s) = L(f(t)).$$

Note. The proof of this theorem is beyond the scope of this book.

Remark. The result of the above theorem can be easily remembered as follows :

If $\mathbf{L(f(t)) = F(s)}$, **then** $\mathbf{L}\left(\frac{\mathbf{f(t)}}{\mathbf{t}}\right) = \int_{\mathbf{s}}^\infty \mathbf{F(S)\, dS}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Laplace transform of the function* $\frac{2}{t}(1-\cos at)$, *for* $t \geq 0$.

Sol. We have $L(2(1 - \cos at)) = 2L(1 - \cos at) = 2[L(1) - L(\cos at)]$

$$= 2\left[\frac{1}{s} - \frac{s}{s^2 + a^2}\right] = \left(\frac{2}{s} - \frac{2s}{s^2 + a^2}\right)^{*}.$$

By **integration of Laplace transform,**

$$L\left(\frac{2(1 - \cos at)}{t}\right) = \int_s^{\infty}\left[\frac{2}{S} - \frac{2S}{S^2 + a^2}\right] dS$$

$$= \left[2 \log S - \log (S^2 + a^2)\right]_s^{\infty} = \log \frac{S^2}{S^2 + a^2}\Bigg|_s^{\infty}$$

$$= \lim_{S \to \infty}\left(\log \frac{S^2}{S^2 + a^2}\right) - \log \frac{s^2}{s^2 + a^2}$$

$$= \log\left(\lim_{S \to \infty} \frac{1}{1 + \dfrac{a^2}{S^2}}\right) - \log \frac{s^2}{s^2 + a^2}$$

$$= \log\left(\frac{1}{1 + 0}\right) - \log \frac{s^2}{s^2 + a^2} = 0 + \log \frac{s^2}{s^2 + a^2} = \log \frac{s^2 + a^2}{s^2}.$$

$$\therefore \qquad L\left(\frac{2}{t}(1 - \cos at)\right) = \mathbf{\log \frac{s^2 + a^2}{s^2}}.$$

Example 2. *Find the Laplace transformation of the function* $\dfrac{2(e^t - \cos t)}{t}$.

Sol. We have $L(2(e^t - \cos t)) = 2L(e^t - \cos t)$

$$= 2\left[\frac{1}{s - 1} - \frac{s}{s^2 + 1}\right] = \frac{2}{s - 1} - \frac{2s}{s^2 + 1}.$$

By **integration of Laplace transform,**

$$L\left(\frac{2(e^t - \cos t)}{t}\right) = \int_s^{\infty}\left(\frac{2}{S - 1} - \frac{2S}{S^2 + 1}\right) dS$$

$$= \left[2 \log (S - 1) - \log (S^2 + 1)\right]\Bigg|_s^{\infty} = \log \frac{(S - 1)^2}{S^2 + 1}\Bigg|_s^{\infty}$$

$$= \lim_{S \to \infty} \log \frac{(S - 1)^2}{S^2 + 1} - \log \frac{(s - 1)^2}{s^2 + 1}$$

$$= \log \lim_{S \to \infty} \frac{(1 - 1/S)^2}{1 + 1/S^2} - \log \frac{(s - 1)^2}{s^2 + 1}$$

***Why this step.** We have not simplified $\dfrac{2}{s} - \dfrac{2s}{s^2 + a^2}$ because we shall be integrating it in the next step.

$$= \log \frac{(1-0)^2}{1+0} - \log \frac{(s-1)^2}{s^2+1} = 0 + \log \frac{s^2+1}{(s-1)^2}$$

$$\therefore \quad L\left(\frac{2(e^t - \cos t)}{t}\right) = \mathbf{\log \frac{s^2+1}{(s-1)^2}}.$$

Example 3. *Find the Laplace transform of the function* $\frac{\sin at}{t}$.

Sol. We have $L(\sin at) = \frac{a}{s^2+a^2} = F(s)$ (say)

By **integration of Laplace transform,**

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty F(S)\, dS.$$

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{S^2+a^2}\, dS = a \cdot \frac{1}{a} \tan^{-1} \frac{S}{a}\Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}.$$

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \mathbf{\cot^{-1}\frac{s}{a}}.$$

Example 4. *Show that :*

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt = \log 3.$$

Sol. $L(e^{-t} - e^{-3t}) = L(e^{-t}) - L(e^{-3t}) = \frac{1}{s+1} - \frac{1}{s+3}$

$\therefore$ *By integration of Laplace transform,*

$$L\left(\frac{e^{-t} - e^{-3t}}{t}\right) = \int_s^\infty \left(\frac{1}{S+1} - \frac{1}{S+3}\right) dS$$

$$= \Big[\log(S+1) - \log(S+3)\Big]_s^\infty = \log \frac{S+1}{S+3}\Big|_s^\infty$$

$$= \lim_{S\to\infty}\left(\log \frac{S+1}{S+3}\right) - \log\frac{s+1}{s+3} = \log\left(\lim_{S\to\infty} \frac{1+\frac{1}{S}}{1+\frac{3}{S}}\right) - \log\frac{s+1}{s+3}$$

$$= \log\left(\frac{1+0}{1+0}\right) + \log\frac{s+3}{s+1} = \log\frac{s+3}{s+1}$$

$$\therefore \quad L\left(\frac{e^{-t} - e^{-3t}}{t}\right) = \log\frac{s+3}{s+1}$$

$$\therefore \quad \int_0^\infty e^{-st} \cdot \frac{e^{-t} - e^{-3t}}{t}\, dt = \log \frac{s+3}{s+1}$$

In particular, let $s = 0$,

$$\therefore \quad \int_0^\infty e^0 \cdot \frac{e^{-t} - e^{-3t}}{t}\, dt = \log \frac{0+3}{0+1}$$

$$\Rightarrow \quad \int_0^\infty \frac{\mathbf{e^{-t} - e^{-3t}}}{\mathbf{t}}\, \mathbf{dt} = \mathbf{log\ 3.}$$

WORKING RULE FOR SOLVING PROBLEMS

Rule If $L(f(t)) = F(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(S)\, dS$.

TEST YOUR KNOWLEDGE

1. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{\sin t}{t}$ (*ii*) $\frac{1 - \cos 2t}{t}$.

2. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{1 - e^t}{t}$ (*ii*) $\frac{e^{-at} - e^{-bt}}{t}$.

3. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{e^{at} - \cos bt}{t}$ (*ii*) $\frac{\cos 2t - \cos 3t}{t}$.

4. Find the Laplace transform of the following functions of t for $t > 0$:

(*i*) $\frac{\cos \alpha t - \cos \beta t}{t}$ (*ii*) $\frac{e^{-t} \sin t}{t}$.

5. Show that the Laplace transform of the function $\frac{\cos at}{t}$ does not exist.

6. Show that :

(*i*) $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$ (*ii*) $\int_0^\infty \frac{e^{-t} \sin t}{t}\, dt = \frac{\pi}{4}$.

(*iii*) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\, dt = \log \frac{2}{3}$.

7. Show that :

(*i*) $L\left(\int_0^t \frac{\sin x}{x}\, dx\right) = \frac{1}{s} \cot^{-1} s$ (*ii*) $L\left(\int_0^t \frac{1 - e^{-2x}}{x}\, dx\right) = \frac{1}{s} \log\left(1 + \frac{2}{s}\right)$.

(*iii*) $L\left(\int_0^t \frac{e^x \sin x}{x}\, dx\right) = \frac{\cot^{-1}(s-1)}{s}$.

Answers

1. (*i*) $\cot^{-1} s$ (*ii*) $\frac{1}{2}\log\frac{s^2+4}{s^2}$ **2.** (*i*) $\log\frac{s-1}{s}$ (*ii*) $\log\frac{s+b}{s+a}$

3. (*i*) $\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}$ (*ii*) $\frac{1}{2}\log\frac{s^2+9}{s^2+4}$ **4.** (*i*) $\frac{1}{2}\log\frac{s^2+\beta^2}{s^2+\alpha^2}$ (*ii*) $\cot^{-1}(s+1)$

Hints

6. We have $L\left(\frac{\sin t}{t}\right) = \cot^{-1} s$

$$\therefore \quad \int_0^\infty \frac{e^{-st}\sin t}{t}\,dt = \cot^{-1} s$$

For part (*i*), take $s = 0$.
For part (*ii*), take $s = 1$.

7. (*i*) $L(\sin t) = \frac{1}{s^2+1} \quad \Rightarrow \quad L\left(\frac{\sin t}{t}\right) = \cot^{-1} s.$

Now using Laplace transform of integral of $\frac{\sin t}{t}$, we have

$$L\left(\int_0^t \frac{\sin T}{T}\,dT\right) = \frac{1}{s}\,.\,\cot^{-1} s.$$