Discrete Structures

Algebraic Structures

- Semi groups
- Monoids
- Groups
- Abelian

Introduction

An algebraic structure is a set equipped with an operation (or operations) that satisfy a standard set of algebraic laws. For example, the set could be the set of all real, natural or integer numbers and the operations could be addition and multiplication (and their inverses, subtraction and division). In this case the standard rules of algebra are:

Introduction

Associative Law of Addition: $(a + b) + c = a + (b + c)$

Commutative Law of Addition: $a + b = b + a$

Additive Identity: $a + 0 = a$

Additive Inverses: $a + -a = 0$

Associative Law of Multiplication: $(a . b) . c = a . (b . c)$

Commutative Law of Multiplication: $a . b = b . a$

Binary operations on a set

Let G be a non empty set

Let G x G : {(a,b)/ a∈G, b∈G}

For example:

If ‘ * ‘ is a binary operation on G, then

For all elements a,b∈ G , a*b∈ G

a*b is called a closure property

ex:

3 and 5 ∈ N

So, 3+5=8 ∈ N

Hence, ‘+’ is also a binary operation for set N.

ex:

7 and 10 ∈ N

7-10 =-3 ∉ N

Hence, ‘-’ is not a binary operation, same way division operation is not a binary operation.

Composition Table

Let S be a finite set, consisting of n elements, then a composition * in S can be described by the table,

Ex:

*	a	b	c
a	a*a	a*b	a*c
b	b*a	b*b	b*c
c	c*a	c*b	c*c

Binary operations on a set

Ex:

Is adition a binary operation on {-1,0,1}?

+	-1	0	1
-1	-2	-1	0
0	-1	0	1
1	0	1	2

Here, addition is not a binary operation, because -2 and 2 both does not belongs to the given set.

Algebraic systems

- **N = {1,2,3,4,…..□ } = Set of all natural numbers.**

  Z = { .....,-4, -3, -2, -1, 0, 1, 2, 3, 4 , ….. □} = Set of all integers.

  Q = Set of all rational numbers.

  R = Set of all real numbers.

- **Binary Operation:** The binary operator * is said to be a binary operation (closed operation) on a non empty set A, if

  a * b ∈ A   for all   a, b ∈ A   (Closure property).

  Ex: The set N is closed with respect to addition and multiplication

  but not w.r.t subtraction and division.

- **Algebraic System:** A set 'A' with one or more binary(closed) operations defined on it is called an algebraic system.

  Ex: (N, +), (Z, +, –), (R, +, . , –) are algebraic systems.

Properties of binary operations

1- Closure:

For all elements a and b ∈ G, a * b ∈ G, then G is closed under '*' operation.

2- Associative:

For all elements a,b,c ∈ G, then

a * (b*c) = (a*b) * c

Ex:

If 2,3,5 ∈ Z, then 2+(3+5) = (2+3)+5

Also 2 . (3 . 5) = (2 . 3) . 5

So, both '+' and '.' are associative

Properties of binary operations

3- Identity:

There exist an element $e \in G$, such that $a * e = a = e * a$, the element is called an identity.

4- Inverse:

An element $b \in G$ is said to be inverse of $a \in G$, such that

$a * a^{-1} = e = a^{-1} * a$

5- Commutative:

For all elements a and $b \in G$ $a * b = b * a$, then '*' is said to be commutative.

Algebraic structures

<u>Groupoid:</u>

A non empty set G with binary operation ‘*’ is called groupoid if the binary operation ‘*’ satisfies the closure property.

<u>Semi Group:</u>

A non empty set G with binary operation ‘*’ is called semigroup if the binary operation ‘*’ satisfies the properties:

- Closure.

- Associative

Algebraic structures

Monoid:

A non empty set G with binary operation ‘*’ is called monoid if the binary operation ‘*’ satisfies the closure properties:

- Closure.

-Associative

-Identity

Group:

A non empty set G with binary operation ‘*’ is called group if the binary operation ‘*’ satisfies the closure properties:

Algebraic structures

-Closure

-Associative

-Identity

-Inverse

Abelian Group:

A non empty set G with binary operation '*' is called abelian if the binary operation '*' satisfies the closure properties:

- Closure.

-Associative

-Identity

-Inverse and Commutative

Algebraic structures

	Closure	Assoc.	Identity	Inverse	Comm.
Abelian Group	Yes	Yes	Yes	Yes	Yes
Group	Yes	Yes	Yes	Yes	No
Monoid	Yes	Yes	Yes	No	No
Semigroup	Yes	Yes	No	No	No

Algebraic structures

Ex:

Let Z be a non empty set and ‘+’ is a binary operation on Z. Is the algebraic system <Z,+> satisfies the abelian group?

1- **Closure property:**

For any two elements a,b such that

$a+b \in Z$

$2+3 = 5 \in Z$

2- **Associative property:**

for any three elements $a,b,c \in Z$, such that

$a + (b + c) = (a + b) + c$

$3 +(4+ 5) = (3 + 4) + 5$

$12 = 12$

Algebraic structures

3- **Identity element:**

There exist a distinguished element e, e ∈ Z , such that

$a + e = a = e + a$

$5 + 0 = 5 = 0 + 5$

So e =0

4- **Inverse property:**

For any element 'a' , a ∈ Z, there exist an element 'a^{-1}' ∈ Z , such that

$a + a^{-1} = a^{-1} + a = e$ or $a + (-a) = (-a) + a = e$

$5 + (-5) = (-5) + 5 = e$

5- **Commutative property:**

$a + b = b + a$

$3 + 4 = 4 + 3$

Algebraic structures

Ex: Let a binary operation '.' on A ={1,3,6,9,12,15,18}. Show that it is an abelian group.

Sol:

1- Closure:

For all elements a,b ∈ A → a.b ∈ A

1 . 3= 3 ∈ A

3 . 6=18 ∈ A

3 . 9 = 27 ∉A

2- Associative:

For all elements a,b,c ∈ A

a * (b*c) = (a*b) * c

1 . (3 . 6) = (1 . 3) . 6

18 = 18

satisfy associative

3- Identity:

a * e = a = e * a

3 . 1 = 3 = 1 . 3

So, e =1 and satisfy the identity property.

Algebraic structures

4- Inverse rpoperty:

$a \,.\, a^{-1} = a^{-1} \,.\, a = e$

$\mathbf{9 \,.\, 9^{-1} = 9^{-1} \,.\, 9 = e = 1}$

5- Commutative property:

$3 \,.\, 6 = 6 \,.\, 3 = 18 \in A$

$1 \,.9 = 9 \,.\, 1 = 9 \quad \in A$

$6 \,.\, 9 = 9 \,.\, 6 = 54 \notin A$

$3 \,.\, 9 = 9 \,.\, 3 = 27 \notin A$

Algebraic structures

Ex: Let a binary operation '-' on a set of integer numbers. Is it satisfy a group or not.

Sol:

1- Closure:

For all elements a,b $\in$ Z $\rightarrow$ a-b $\in$ Z

1 - 3= -2 $\in$ Z

3 - 6= -3 $\in$ Z

9 - 3 = 6 $\in$ Z

Algebraic structures

2- Associative:

For all elements a,b,c ∈ A

a - (b-c) = (a-b) - c

1 - (3 - 6) = (1 - 3) - 6

4 = -8

not satisfy associative

3- Identity:

a - e = a = e - a

3 - 0 = 3 = 0 - 3 = -3

So not satisfy the identity property.

Semigroup

- Semigroup: An algebraic system (S,*) is called an algebraic system if it is satisfy the following property:
- a*(b*c) = (a*b)*c
- Ex: (N,+)   5+10=15
- 5+(10+15) = (5+10)+15
- 30 = 30
- (Z,-)   5-(10-15) = (5-10)-15
- 10 = -20
- (Z,.) 5.(2.1) = (5.2) .1
- 10 = 10
- $2^5.(2^{10}.2^{15}) = (2^5 .2^{10}).2^{15}$
- $2^5*2^{25} = 2^{15}*2^{15}$
- $2^{30} = 2^{30}$