Tangents Plane and Normal Lines on the Surface
المستويات المماسة والمستقيمات العمودية على السطح

Definition: Let $f(x,y,z)=0$ define a surface that is differentiable at a point $p(x_0,y_0,z_0)$, then the tangent plane to $f(x,y,z)$ at $p(x_0,y_0,z_0)$ is the plane with normal vector (non-zero gradient vector) that passes through the point $p(x_0,y_0,z_0)$. In Particular the equation of the tangent plane is

$$\boxed{f_x\,(x-x_o)+f_y\,(y-y_o)+f_z\,(z-z_o)=0}$$

Let $f(x,y,z)=0$ define a surface that is differentiable at a point $p(x_0,y_0,z_0)$,then the normal line to $f(x,y,z)$ at $p(x_0,y_0,z_0)$ is the line with normal vector (non-zero gradient vector) that passes through the point $p(x_0,y_0,z_0)$. In Particular the equation of the normal line is

$$\boxed{\frac{x-x_o}{f_x}=\frac{y-y_o}{f_y}=\frac{z-z_o}{f_z}}$$

Example1: Find the tangent plane and the normal line of the surface

$$f(x,y,z)=x^2+y^2+z-9 \quad \text{at} \quad p(1,2,4).$$

Solution:

$$\nabla f(x_0,y_0,z_0)=f_x(x_0,y_0,z_0)i+f_y(x_0,y_0,z_0)j+f_z(x_0,y_0,z_0)k$$

$$f_x(x,y,z)=2x \quad \Rrightarrow \quad f_x(1,2,4)=2$$
$$f_y(x,y,z)=2y \quad \Rrightarrow \quad f_y(1,2,4)=4$$
$$f_z(x,y,z)=1 \quad \Rrightarrow \quad f_z(1,2,4)=1$$

$\nabla f(1,2,4)=2i+4j+k$. So, ∇f is non-zero vector.

The tangent plane is

$$f_x\,(x-x_o)+f_y\,(y-y_o)+f_z\,(z-z_o)=0$$
$$2\,(x-1)+4\,(y-2)+\,(z-4)=0$$
$$2x+4y+z=14$$

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And the normal line

$$\frac{x-x_o}{f_x}=\frac{y-y_o}{f_y}=\frac{z-z_o}{f_z}$$
$$\frac{x-1}{2}=\frac{y-2}{4}=z-4$$

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Example2: Find the tangent plane and the normal line of the surface

$$4x^2 - 9y^2 - 9z^2 - 36 = 0 \quad \text{at} \quad p(3\sqrt{3}, 2,2).$$

Solution: let $f(x, y, z) = 4x^2 - 9y^2 - 9z^2 - 36$

$$f_x(x, y, z) = 8x \quad \Rrightarrow \quad f_x(3\sqrt{3}, 2,2) = 24\sqrt{3}$$
$$f_y(x, y, z) = -18y \quad \Rrightarrow \quad f_y(3\sqrt{3}, 2,2) = -36$$
$$f_z(x, y, z) = -18z \quad \Rrightarrow \quad f_z(3\sqrt{3}, 2,2) = -36$$

$\nabla f(3\sqrt{3}, 2,2) = 24\sqrt{3}i - 36j - 36k$. So, ∇f is non-zero vector.

The tangent plane is

$$f_x\,(x - x_o) + f_y\,(y - y_o) + f_z\,(z - z_o) = 0$$
$$24\sqrt{3}\,(x - 3\sqrt{3}) - 36\,(y - 2) - 36\,(z - 2) = 0$$
$$2\sqrt{3}\,x - 3y - 3z = 6$$

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And the normal line

$$\frac{x - x_o}{f_x} = \frac{y - y_o}{f_y} = \frac{z - z_o}{f_z}$$

$$\frac{x-3\sqrt{3}}{2\sqrt{3}} = \frac{y-2}{-3} = \frac{z-2}{-3}$$

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Example3: Find the tangent plane and the normal line of the surface

$$f(x, y, z) = z + \ln(x^2 + y^2) \quad \text{at} \quad p(1,0,0).$$

Solution:

$$f_x(x, y, z) = \frac{2x}{x^2+y^2} \quad \Rrightarrow \quad f_x(1,0,0) = 2$$
$$f_y(x, y, z) = \frac{2y}{x^2+y^2} \quad \Rrightarrow \quad f_y(1,0,0) = 0$$
$$f_z(x, y, z) = 1 \quad \Rrightarrow \quad f_z(1,0,0) = 1$$

$\nabla f(1,0,0) = 2i + k$. So, ∇f is non-zero vector.

The tangent plane is

$$f_x\,(x - x_o) + f_y\,(y - y_o) + f_z\,(z - z_o) = 0$$
$$2\,(x - 1) + z = 0$$
$$2\,x + z = 2$$

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And the normal line

$$\frac{x - x_o}{f_x} = \frac{y - y_o}{f_y} = \frac{z - z_o}{f_z}$$

$$\frac{x-1}{2} = z \quad , \quad y = 0$$

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Example 4: Find the tangent plane and the normal line of the surface

$$x^2 - xy - y^2 - z = 0 \quad \text{at} \quad p(1,1,-1).$$

Solution: let $f(x,y,z) = x^2 - xy - y^2 - z$

$$f_x(x,y,z) = 2x - y \qquad \Rrightarrow \qquad f_x(1,1,-1) = 1$$
$$f_y(x,y,z) = -x - 2y \qquad \Rrightarrow \qquad f_y(1,1,-1) = -3$$
$$f_z(x,y,z) = -1 \qquad \Rrightarrow \qquad f_z(1,1,-1) = -1$$

$\nabla f(1,1,-1) = i - 3j - k$. So, ∇f is non-zero vector.

The tangent plane is

$$f_x\,(x - x_o) + f_y\,(y - y_o) + f_z\,(z - z_o) = 0$$
$$(x - 1) - 3(y - 1) - (z + 1) = 0$$
$$x - 3y - z = -1$$

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And the normal line

$$\frac{x - x_o}{f_x} = \frac{y - y_o}{f_y} = \frac{z - z_o}{f_z}$$
$$x - 1 = \frac{y-1}{-3} = -(\,z + 1)$$

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Example 5: Find the tangent plane and the normal line of the surface

$$f(x,y,z) = 2z - x^2 \quad \text{at} \quad p(2,0,2).$$

Solution:

$$f_x(x,y,z) = -2x \quad \Rrightarrow \quad f_x(2,0,2) = -4$$
$$f_y(x,y,z) = 0 \quad \Rrightarrow \quad f_y(2,0,2) = 0$$
$$f_z(x,y,z) = 2 \quad \Rrightarrow \quad f_z(2,0,2) = 2$$

$\nabla f(2,0,2) = -4i + 2k$. So, ∇f is non-zero vector.

The tangent plane is

$$f_x\,(x - x_o) + f_y\,(y - y_o) + f_z\,(z - z_o) = 0$$
$$-4\,(x - 2) + \ 2(z - 2) = 0$$
$$2\,x - z = 2$$

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And the normal line

$$\frac{x - x_o}{f_x} = \frac{y - y_o}{f_y} = \frac{z - z_o}{f_z}$$
$$\frac{x - 2}{2} = 2 - \ z \quad , \quad y = 0$$

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