Ex $f(x) = \sqrt{x}$

Solu $f(x+h) = \sqrt{x+h}$

$$\because f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

**Ex** $f(x) = \sin x$

$$f(x+h) = \sin(x+h)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2} \sin\frac{A-B}{2}$$

$$\sin(x+h) - \sin x = 2\cos\frac{x+h+x}{2} \sin\frac{x+h-x}{2}$$

$$= 2\cos x\frac{2x+h}{2} \sin\frac{h}{2}$$

$$\therefore f'(x) = \lim_{h\to 0} \frac{2\cos x\frac{2x+h}{2} \sin\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} 2\cos x\frac{2x+h}{2} \cdot \lim_{h\to 0} \frac{\sin\frac{h}{2}}{h}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad ; \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\therefore f'(x) = 2\cos\frac{2x}{2} \cdot 1$$

$$= \frac{1}{2} 2\cos 2x$$

$$= \cos x \qquad ?$$

$f(x) = \sin x$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(x+h) = \sin x \cosh + \cos x \sinh$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \sinh}{h} + \lim_{h \to 0} \frac{\sin x \cosh - \sin x}{h}$$

$$= \cos x \lim_{h \to 0} \frac{\sinh}{h} + \lim_{h \to 0} \frac{\sin x(\cosh - 1)}{h}$$

$$= \cos x \lim_{h \to 0} \frac{\sinh}{h} + \sin x \lim_{h \to 0} \frac{\cosh - 1}{h}$$

But $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

$$f'(x) = \cos x \cdot 1 + \sin x \cdot 0$$

$$= \cos x$$

Ex $f(x) = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{\cos A \cos B - \sin A \sin B - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x (\cosh - 1)}{h} - \lim_{h \to 0} \sin x \frac{\sinh}{h}$$

$$= \cos x \lim_{h \to 0} \frac{\cosh - 1}{h} - \sin x \lim_{h \to 0} \frac{\sinh}{h}$$

But $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

$$\therefore f'(x) = \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x$$

# Implicit Differentiation

Def.: We will Say that agiven equation in x and y defines the function f implicity y if the graph of $y = f(x)$ Coincides with a partion of the graph of the equation

Ex Use implicit differentiation to find $\frac{dy}{dx}$ if:

1. $x^2 + y^2 = 1$

Solu $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$

2. $y^2 - 5xy = 6$

$$2y\frac{dy}{dx} - 5[x\frac{dy}{dx} + y \cdot 1] = 0$$

$$2y\frac{dy}{dx} - 5x\frac{dy}{dx} - 5y = 0$$

$$\frac{dy}{dx}[2y - 5x] = +5y \Rightarrow \frac{dy}{dx} = \frac{5y}{2y - 5x}$$

3. $5y^2 + \sin y = x^2$

$$10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}(10y + \cos y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

4. $x\cos y = y$

Sol:
$$x\left(-\sin y\frac{dy}{dx} + \cos y \cdot 1\right) = \frac{dy}{dx}$$

$$-x\sin y\frac{dy}{dx} + x\cos y = \frac{dy}{dx}$$

$$\cos y = \frac{dy}{dx}(1 + x\sin y)$$

$$\frac{dy}{dx} = \frac{\cos y}{1 + x\sin y}$$

H.W. Using implicit differentiation to find $\frac{dy}{dx}$ for:-

1. $x^2 = \dfrac{\cot y}{1+\csc y}$

2. $\sin(x^2y^2) = x$

3. $\tan^3(xy^2 + y) = x$

4. $x^3 - y^3 = 6xy$

5. $y + \sin y = x$

6. $x^3y^3 - 4e^x = 0$

7. $\dfrac{1}{y} + \dfrac{1}{x} = 1$

**Ex** 5. $x^2+y^2+2xy=4$

**Solu**

$$2x+2y\frac{dy}{dx}+2\left[x\frac{dy}{dx}+y\cdot 1\right]=0$$

$$2x+2y\frac{dy}{dx}+2x\frac{dy}{dx}+2y=0$$

$$\frac{dy}{dx}\left[2y+2x\right]=-2x-2y$$

$$\frac{dy}{dx}=\frac{-2(x+y)}{2(y+x)}$$

$$\frac{dy}{dx}=-1$$

6. $y=2x+\sin(y-2x)$

**Solu**

$$\frac{dy}{dx}=2+\cos(y-2x)\cdot\left(\frac{dy}{dx}-2\right)$$

$$\frac{dy}{dx}=2+\cos(y-2x)\frac{dy}{dx}-2\cos(y-2x)$$

$$2\cos(y-2x)-2=\cos(y-2x)\frac{dy}{dx}-\frac{dy}{dx}$$

$$2\left[\cos(y-2x)-1\right]=\frac{dy}{dx}\left(\cos(y-2x)-1\right)$$

$$\frac{dy}{dx}=\frac{2\left(\cos(y-2x)-1\right)}{\left(\cos(y-2x)-1\right)}$$

$$\frac{dy}{dx}=2$$

6. $e^x + \cos y - 2 = 0$

Find $\frac{dy}{dx}$ at the point $(0, \frac{\pi}{3})$

Solu

$e^x + (-\sin y \frac{dy}{dx}) = 0$

$e^x - \sin y \frac{dy}{dx} = 0$

$e^x = \sin y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{e^x}{\sin y}$ at the point $(0, \frac{\pi}{3})$

$\frac{dy}{dx} = \frac{e^0}{\sin \frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

7. $y = \sin^3 x^3 \Rightarrow y = (\sin x^3)^3$

$y' = 3(\sin x^3)^2 \cdot \cos x^3 \cdot 3x^2$

$= 9x^2 \cos x^3 \sin^3 x^3$

8. $y = e^{1/x}$

$y' = e^{\cancel{x^{-1}}} \cdot \frac{-1}{x^2}$

9. $y = \sqrt{x}\, e^{-\sin x}$

$y = \sqrt{x} \frac{1}{e^{\sin x}} = \frac{x^{\frac{1}{2}}}{e^{\sin x}}$