5^{th} Lecture

Interior Set   المجموعة الداخلية

Definition (1.6): Let (X, τ) be a topological space and $E \subset X$, we define the **interior** of a set E, denoted by E° or $i(E)$ as follow:

$$E^\circ = \bigcup_{\forall\, G \subset E} G, \text{ where } G \text{ is open}$$

Example (1.10):

Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$.

Find the interior of the following sets

$$A = \{a, b, e\},\ B = \{a, b, c\},\ C = \{c\} \text{ and } E = \{a, c, d\}$$

Solution:

$A = \{a, b, e\}$

The open sets contained in A are $\emptyset$, $\{a\}$,

$A^\circ = \bigcup_{\forall\, G \subset A} G$, where G is open

$\Rightarrow\ A^\circ = \emptyset \cup \{a\} = \{a\}$

Now $B = \{a, b, c\}$

The open sets contained in B are $\emptyset$, $\{a\}$,

$B^\circ = \bigcup_{\forall\, G \subset B} G$, where G is open

$\Rightarrow\ B^\circ = \emptyset \cup \{a\} = \{a\}$

Now $C = \{c\}$

The open sets contained in C is $\emptyset$

$C^\circ = \bigcup_{\forall\, G \subset C} G$, where G is open

$\Rightarrow\ C^\circ = \cup \emptyset = \emptyset$

Now $E = \{a, c, d\}$

The open sets contained in E are $\emptyset, \{a\}, \{c, d\}, \{a, c, d\}$

$E^\circ = \bigcup_{\forall\, G \subset E} G$, where G is open

$\Rightarrow\ E^\circ = \emptyset \cup \{a\} \cup \{c, d\} \cup \{a, c, d\} = \{a, c, d\}$

Remark (1.7): If is possible to find the interior of a set if its closure is given as in the following theorem:

Theorem (1.5): For every subset E of (X, τ) we have

$$E^\circ = \overline{E^c}^{\,c}$$

Proof:

Let $x \in E^\circ \ \Rightarrow\ x \notin E^c, \quad \forall\, x \in E^\circ$

Now $E^\circ \cap E^c = \emptyset$

$\Rightarrow\ (E^c \cap E^\circ) \backslash \{x\} = \emptyset$

$\Rightarrow\ x \notin d(E^c)$ also $x \notin E^c$

$\Rightarrow\ x \notin E^c \cup d(E^c)$

$\Rightarrow\ x \notin \overline{E^c}$

$\Rightarrow\ x \in \overline{E^c}^{\,c}$

$\Rightarrow\ E^\circ \subset \overline{E^c}^{\,c}$ ……………….. (1)

Now, let $x \in \overline{E^c}^{\,c}$

$\Rightarrow\ x \notin \overline{E^c}$

$\Rightarrow\ x \notin E^c \cup d(E^c)$

$\Rightarrow\ x \notin E^c \wedge x \notin d(E^c)$

Since $x \notin d(E^c) \ \Rightarrow\ x$ is not a limit point of E^c

$\Rightarrow\ \exists\ open\ G_x \ni x; (E^c \cap G_x) \backslash \{x\} = \emptyset$

$\Rightarrow\ \exists\ open\ G_x \ni x; E^c \cap G_x = \emptyset$

$\Rightarrow\ \exists\ open\ G_x \ni x; G_x \subset E$

$\Rightarrow\ x \in E^\circ$

$\Rightarrow\ \overline{E^c}^c \subset E^\circ$ ……………….. (2)

From (1) and (2) we get

$$E^\circ = \overline{E^c}^c$$

Theorem (1.6): If A, B are subsets of (X, τ). Then

(i) $X^\circ = X,\ \emptyset^\circ = \emptyset$

(ii) A° is the largest open set in A

(iii) $A^\circ \subset A$

(iv) $A \subset B\ \Rightarrow\ A^\circ \subset B^\circ$

(v) $A^{\circ^\circ} = A^\circ$

(vi) $(A \cap B)^\circ = A^\circ \cap B^\circ$

Proof:

(i) $X^\circ = \overline{X^c}^c$

$= \bar{\emptyset}^c = \emptyset^c = X$

$\emptyset^\circ = \overline{\emptyset^c}^c$

$= \bar{X}^c = X^c = \emptyset$

(ii) $A^\circ = \bigcup_{\alpha\in\Lambda} G_\alpha\ ,\ \forall\ open\ G_\alpha \subset A$

$\Rightarrow\ \bigcup_{\alpha\in\Lambda} G_\alpha \subset A$

Since $\bigcup_{\alpha\in\Lambda} G_\alpha \supset G_\alpha, \alpha \in \Lambda$

$\Rightarrow\ A^\circ$ is the largest open set in A

(iii) From (ii) we have $A^\circ \subset A$

(iv) $A^\circ = \bigcup_{\forall\, G_\alpha \subset A \subset B} G_\alpha \text{ , } \forall\ open\ G_\alpha \subset B$

$A^\circ \subset B^\circ$

(v) $(A^\circ)^\circ = \overline{\left(\overline{A^c}^c\right)^c}^c = \overline{(\overline{A^c})}^c = (\overline{A^c})^c = A^\circ$

(vi) We have

$A \cap B \subset A \;\; \Rightarrow \;\; (A \cap B)^\circ \subset A^\circ$

$A \cap B \subset B \;\; \Rightarrow \;\; (A \cap B)^\circ \subset B^\circ$

$\Rightarrow \;\; (A \cap B)^\circ \subset (A^\circ \cap B^\circ)$ ……………… (1)

We need to show that $(A^\circ \cap B^\circ) \subset (A \cap B)^\circ$

Let $x \notin (A \cap B)^\circ$

$\Rightarrow$ x is not an interior point of $(A \cap B)$

$\Rightarrow \; \forall\ open\ G_x \ni x, x \in G_x \not\subset (A \cap B)$

$\Rightarrow \; \forall\ open\ G_x \ni x, x \in G_x \not\subset A \; \vee \; \forall\ open\ G_x \ni x, x \in G_x \not\subset B$

$\Rightarrow \;\; x \notin A^\circ \; \vee \; x \notin B^\circ$

$\Rightarrow \;\; x \notin A^\circ \cap B^\circ$

$\Rightarrow \;\; (A^\circ \cap B^\circ) \subset (A \cap B)^\circ$ ……………… (2)

From (1) and (2) we get

$$(A \cap B)^\circ = A^\circ \cap B^\circ$$

Exercises (1.5): (Homework)

(1) Prove that $\bar{E} = E^{c^{\circ c}}$.

(2) Disprove that $(A \cup B)^\circ = A^\circ \cup B^\circ$. (Give an example)
