5) Integration by Substitution with trigonometric Functions:-

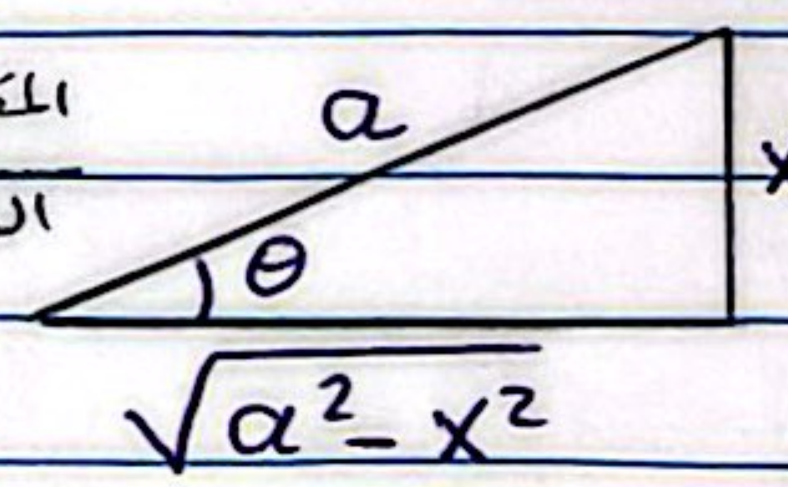
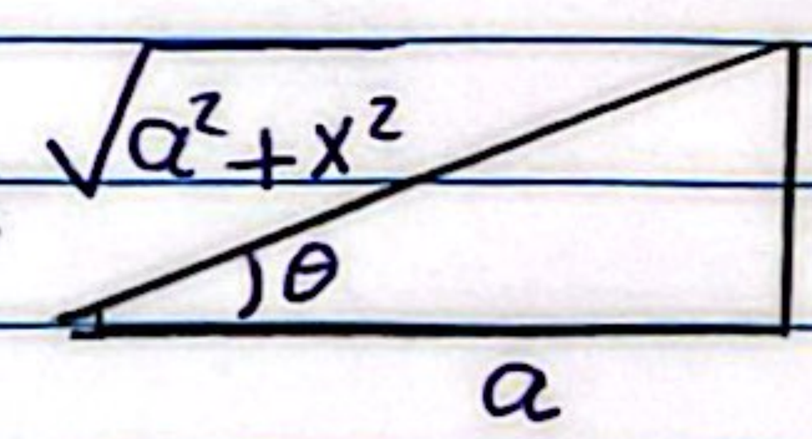
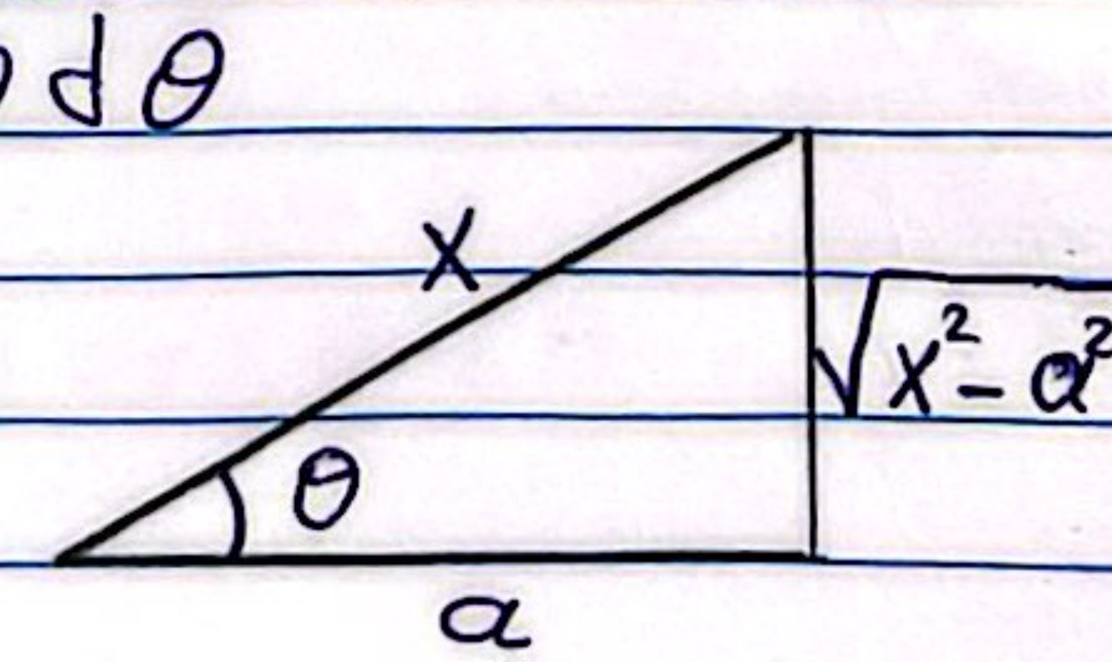
Sometimes the function to be integrated can be converted into a trigonometric function that can be integrated.

This integration is called trigonometric substitution

This method is used if the integration contains:-

$$\sqrt{a^2-x^2} \quad \text{or} \quad \sqrt{a^2+x^2} \quad \text{or} \quad \sqrt{x^2-a^2}$$

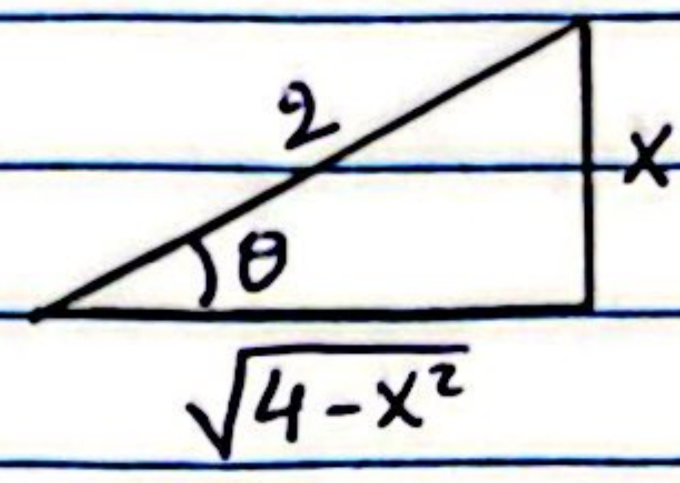
To find this integration, we use substitutions as shown in the following table:-

The amount	Compensation	shape
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ $\frac{\text{المقابل}}{\text{الوتر}}$ $dx = a\cos\theta\, d\theta$	
$\sqrt{a^2+x^2}$	$x = a\tan\theta$ $\frac{\text{المقابل}}{\text{المجاور}}$ $dx = a\sec^2\theta\, d\theta$	
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ $\frac{\text{الوتر}}{\text{المجاور}}$ $dx = a\sec\theta\tan\theta\, d\theta$	

Ex Find the following integrals :-

1. $\int \frac{dx}{\sqrt{4-x^2}}$

Solu let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta \, d\theta$



$$\therefore \int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta \, d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta \, d\theta}{\sqrt{4(\underbrace{1-\sin^2\theta}_{\cos^2 x})}} = \int \frac{2\cos\theta \, d\theta}{2\cos\theta}$$

$$= \int d\theta = \theta + c$$

$$x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}\frac{x}{2}$$

$$\therefore \int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + c$$

2. $\int \sqrt{4-x^2}\, dx$

Solu let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta \, d\theta$

$$\therefore \int \sqrt{4-x^2}\, dx = \int \sqrt{4-(2\sin\theta)^2}\; 2\cos\theta \, d\theta$$

$$= \int \sqrt{4(\underbrace{1-\sin^2\theta}_{\cos^2\theta})} \cdot 2\cos\theta \, d\theta$$

$$= \int 2\cos\theta \cdot 2\cos\theta \, d\theta = \int 4\cos^2\theta \, d\theta$$

$$\because \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$= 4\int \frac{1}{2}(1+\cos 2\theta) d\theta$$

$$= 2\left[\int d\theta + \int \cos 2\theta \, d\theta\right]$$

$$= 2\left(\theta + \frac{\sin 2\theta}{2}\right) + c = 2\theta + \sin 2\theta + c$$

$$\because x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2} \Rightarrow \theta = \sin^{-1}\frac{x}{2}$$

and $\sin 2\theta = 2\sin\theta\cos\theta$

المقابل / الوتر المجاور / الوتر

$$\therefore \int \sqrt{4-x^2}\, dx = 2\sin^{-1}\frac{x}{2} + \sin\theta\cos\theta + c$$

$$= 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\,\frac{\sqrt{4-x^2}}{2} + c$$

3. $\int \frac{dx}{x^2\sqrt{x^2+9}}$

Solu

let $x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta \, d\theta$

$\because \int \frac{dx}{x^2\sqrt{x^2+9}} = \int \frac{3\sec^2\theta \, d\theta}{9\tan^2\theta\sqrt{9\tan^2\theta+9}}$

$= \int \frac{3\sec^2\theta \, d\theta}{9\tan^2\theta\sqrt{9(\underbrace{\tan^2\theta+1}_{\sec^2\theta})}}$

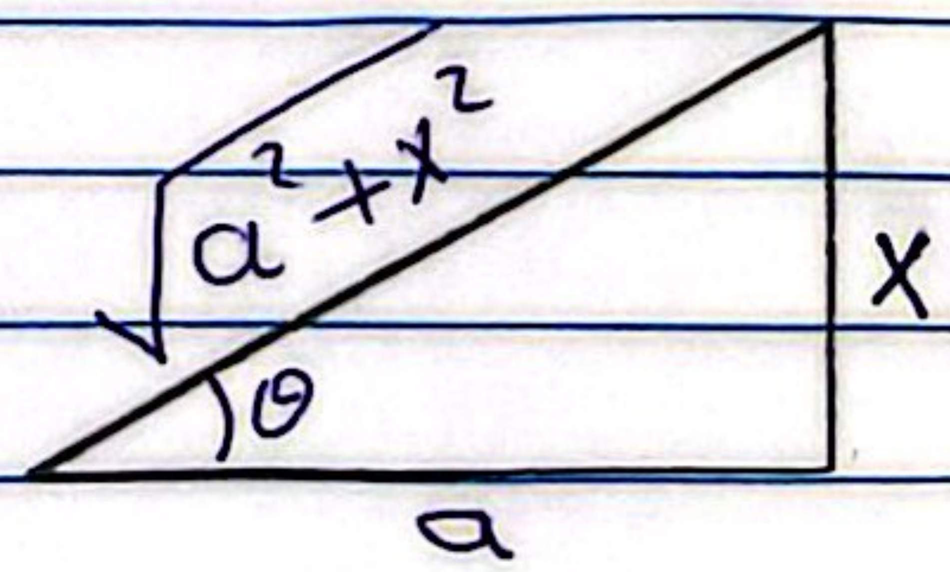
$= \int \frac{\cancel{3}\sec^{\cancel{2}}\theta \, d\theta}{9\tan^2\theta \, . \, \cancel{3}\cancel{\sec\theta}} = \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta} d\theta$

$= \frac{1}{9}\int \left(\overset{\sec}{\frac{1}{\cancel{\cos\theta}}} \cdot \overset{\frac{1}{\tan^2}}{\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}}\right) d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta} d\theta$

$= \frac{1}{9}\int \cos\theta \sin^{-2}\theta \, d\theta = \frac{1}{9}\frac{(\sin\theta)^{-2+1}}{-2+1}$

$= \frac{-1}{9}(\sin\theta)^{-1} = \frac{-1}{9\sin\theta} + c$

$\sin\theta = \frac{x}{\sqrt{9+x^2}}$



$\because \int \frac{dx}{x^2\sqrt{x^2+9}} = \frac{-\sqrt{9+x^2}}{9x} + c$

4. $\displaystyle\int \frac{dx}{\sqrt{x^2-9}}$

Solu

let $x = a\sec\theta \Rightarrow x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\, d\theta$

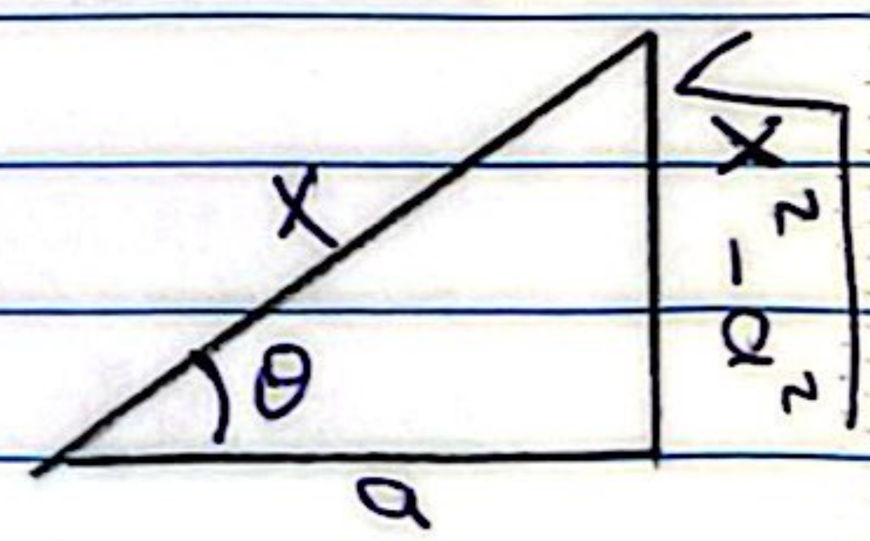
$$\therefore \int \frac{dx}{\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}}$$

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9(\underbrace{\sec^2\theta - 1}_{\tan^2\theta})}} = \int \frac{\cancel{3}\sec\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{3\tan\theta}}$$

$$= \int \sec\theta\, d\theta * \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$\tan\theta = \dfrac{\sqrt{x^2 - a^2}}{a}$ $\quad \dfrac{\text{المقابل}}{\text{المجاور}}$



$= \dfrac{\sqrt{x^2-9}}{3}$; $x = 3\sec\theta \Rightarrow \sec\theta = \dfrac{x}{3}$ $\quad \dfrac{\text{المجاور}}{\text{الوتر}}$

$$\therefore \int \frac{dx}{\sqrt{x^2-9}} = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$

H.w.

5. $$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx$$

Sol/

$$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx = \int \frac{16}{x^2\sqrt{\frac{4}{9}-x^2}}\,dx \qquad ;\ a=\frac{2}{3}$$

let $x = a\sin\theta \Rightarrow x=\frac{2}{3}\sin\theta\ ;\ dx=\frac{2}{3}\cos\theta\,d\theta$

or $$\int \frac{16}{x^2\sqrt{4-9x^2}}\,dx = \int \frac{16}{x^2\sqrt{\frac{9\times4}{9}-9x^2}}\,dx$$

$$= \int \frac{16}{3x^2\sqrt{\frac{4}{9}-x^2}}\,dx$$

باستخدام الفرضية في اعلاه نحصل cos :

$$= \frac{16}{3}\int \frac{\frac{2}{3}\cos\theta\,d\theta}{\frac{4}{9}\sin^2\theta\sqrt{\frac{4}{9}-\frac{4}{9}\sin^2\theta}} = 8\int \frac{\cos\theta\,d\theta}{\sin^2\theta\sqrt{\frac{4}{9}-\frac{4}{9}\sin^2\theta}}$$

$$= 8\int \frac{\cos\theta\,d\theta}{\frac{2}{3}\sin^2\theta\underbrace{\sqrt{1-\sin^2\theta}}_{\cos\theta}} \overset{\frac{3}{2}\times 8}{=} 12\int \frac{\cos\theta\,d\theta}{\sin^2\theta\cos\theta}$$

$$= 12\int \frac{d\theta}{\sin^2\theta} = 12\int \csc^2\theta\,d\theta = -12\cot\theta + c$$

$$\therefore \int \frac{16}{x^2\sqrt{4-9x^2}}\,dx = \frac{-12\sqrt{\frac{4}{9}-x^2}}{x} + c$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\frac{\text{المقابل}}{\text{المجاور}}} = \frac{\text{المجاور}}{\text{المقابل}}$$

6. $\int \sqrt{9-4x^2}\, dx$ نجعل معامل x^2 داخل الجذر يساوي ١

Solu

$$\int \sqrt{\frac{9\times 4}{9} - 4x^2}\, dx = 2\int \sqrt{\frac{9}{4} - x^2}\, dx$$

let $x = a\sin\theta \Rightarrow x = \frac{3}{2}\sin\theta \;;\; dx = \frac{3}{2}\cos\theta\, d\theta$

$$\Rightarrow 2\int \sqrt{\frac{9}{4} - x^2}\, dx = 2\int \sqrt{\frac{9}{4} - \frac{9}{4}\sin^2\theta} \cdot \frac{3}{2}\cos\theta\, d\theta$$

$$= 2\,\frac{3}{2}\,\frac{3}{2}\int \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= \frac{9}{2}\int \cos\theta\cos\theta\, d\theta = \frac{9}{2}\int \cos^2\theta\, d\theta$$

$$= \frac{9}{2}\int \frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= \frac{9}{4}\left(\int d\theta + \int \cos(2\theta)\, d\theta\right) = \frac{9}{4}\left(\int\theta + \frac{1}{2}\int \cos 2\theta\, d\theta\right)$$

$$= \frac{9}{4}\left(\theta + \frac{\sin(2\theta)}{2}\right) + c$$

$$x = \frac{3}{2}\sin\theta \Rightarrow \frac{2}{3}x = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{2}{3}x\right)$$

$$\therefore \sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{x}{\frac{3}{2}} , \frac{\sqrt{\frac{9}{4}-x^2}}{\frac{3}{2}}\right)$$

$$= \frac{8}{9}x\sqrt{\frac{9}{4}-x^2}$$

$$\therefore \int \sqrt{9-4x^2}\, dx = \frac{9}{4}\left(\sin^{-1}\left(\frac{2}{3}x\right) + \frac{\frac{8}{9}x + \sqrt{\frac{9}{4}-x^2}}{2}\right) + c$$

$$= \frac{9}{4}\sin^{-1}\left(\frac{2}{3}x\right) + x\sqrt{\frac{9}{4}-x^2} + c$$

H.w.

Find the integral Substitutation using trigonometric function :-

1. $\int \frac{\sqrt{x^2-3}}{x}\, dx$

Solu

let $x = a\sec\theta \Rightarrow x = \sqrt{3}\sec\theta \; ; \; dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$

$$\therefore \int \frac{\sqrt{x^2-3}}{x}\, dx = \int \frac{\sqrt{(\sqrt{3}\sec\theta)^2-3}}{\cancel{\sqrt{3}\sec\theta}} \cdot \cancel{\sqrt{3}\sec\theta}\tan\theta\, d\theta$$

$$= \int \sqrt{3\sec^2\theta - 3} \cdot \tan\theta\, d\theta$$

$$= \int \sqrt{3(\underbrace{\sec^2\theta - 1}_{\tan^2\theta})} \cdot \tan\theta\, d\theta$$

$$= \int \sqrt{3}\tan\theta\tan\theta\, d\theta = \sqrt{3}\int \tan^2\theta\, d\theta$$

$$= \sqrt{3}\int (\sec^2\theta - 1)\, d\theta$$

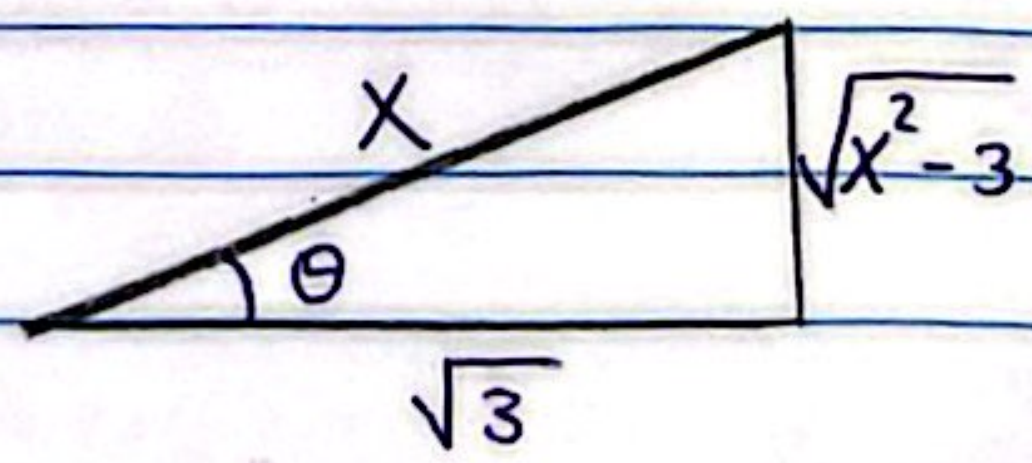
$$= \sqrt{3}(\tan\theta - \theta) + c$$

$$= \sqrt{3}\left(\frac{\sqrt{x^2-3}}{\sqrt{3}} - \sec^{-1}\frac{x}{\sqrt{3}}\right) + c$$

$x = \sqrt{3}\sec\theta \Rightarrow \frac{x}{\sqrt{3}} = \sec\theta \Rightarrow \theta = \sec^{-1}\frac{x}{\sqrt{3}}$

$\frac{\text{الوتر}}{\text{المجاور}}$

$\tan \frac{\text{المقابل}}{\text{المجاور}}$



2. $$\int \frac{dx}{(\sqrt{9-x^2})^3} = \int \frac{dx}{(9-x^2)^{3/2}}$$

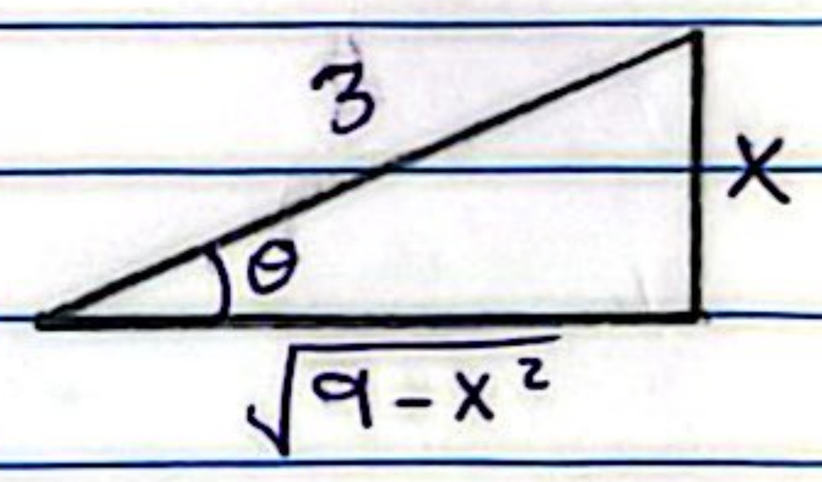
Solu $x = a\sin\theta$; $x = 3\sin\theta$; $dx = 3\cos\theta\, d\theta$

$$= \int \frac{3\cos\theta\, d\theta}{(\sqrt{(9-9\sin^2\theta)})^3} = \int \frac{3\cos\theta}{(3\sqrt{\cos^2\theta})^3}\, d\theta$$

$$= \int \frac{3\cos\theta}{27\cos^3\theta}\, d\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \sec^2\theta\, d\theta = \frac{1}{9}\tan\theta + c \quad \frac{\text{المقابل}}{\text{المجاور}}$$

$x = 3\sin\theta \Rightarrow \sin\theta = \dfrac{x}{3}$ $\quad \dfrac{\text{المقابل}}{\text{الوتر}}$



$$\therefore \int \frac{dx}{(\sqrt{9-x^2})^3} = \frac{1}{9}\,\frac{x}{\sqrt{9-x^2}} + c$$

3. $$\int \frac{\sqrt{x^2-16}}{x^2}$$

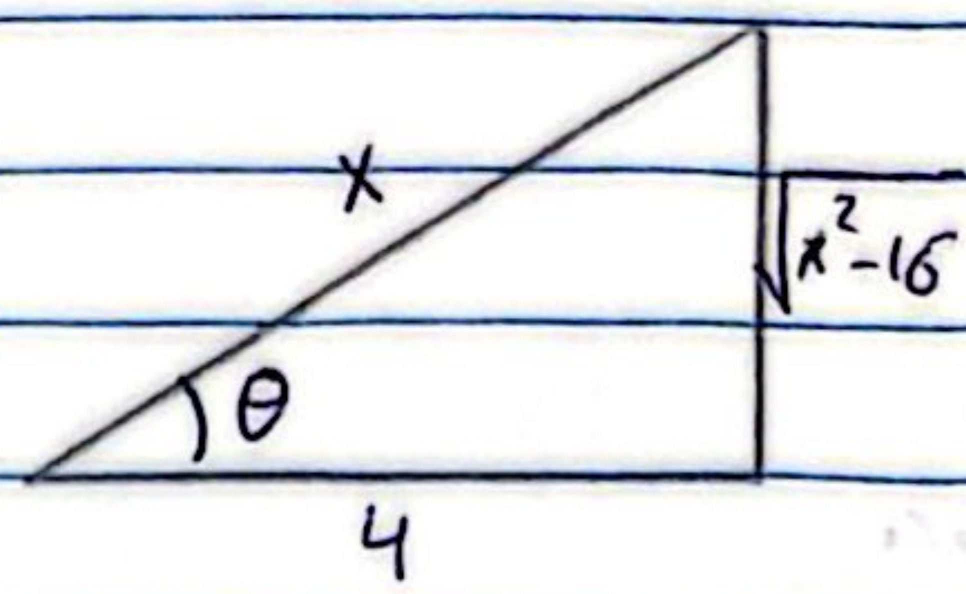
Solu

let $x = 4\sec\theta$; $d\theta = 4\sec\theta\tan\theta\, d\theta$

$$\therefore \int \frac{\sqrt{x^2-16}}{x^2}\, dx = \int \frac{\sqrt{16\sec^2\theta - 16}}{\underset{4}{16}\sec^2\theta}\, 4\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{4\tan\theta}{4\sec\theta} \cdot \tan\theta\, d\theta = \int \frac{\tan^2\theta}{\sec\theta}\, d\theta$$

$\because \tan^2\theta = \sec^2\theta - 1$



$\therefore \int \frac{\tan^2\theta}{\sec\theta}\, d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta$

$x = 4\sec\theta \Rightarrow \sec\theta = \frac{x}{4}$

$= \int (\sec\theta - \cos\theta)\, d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta$

$= \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| - \frac{\sqrt{x^2-16}}{4} + c$