2.1 Linear operators

In Calculus we have functions, which is a mapping between to subset from the field of real number. In vector space The mapping between them is called an operator.

Definition Operator

Let X, Y be two vectors spaces The operator is a mapping T from X into Y denoted by $T: X \longrightarrow Y$ such that $T(x) = y$ where $x \in X$, $y \in Y$

Example Let $X = R^2$ and $Y = R$ and $T: R^2 \longrightarrow R$ such that

$T(x) = T(x_1, x_2) = x_1$ is operator

$S: R^2 \longrightarrow R$ $x \in R^2$, $x_1, x_2 \in R$

$S(x) = S(x_1, x_2) = x_2$ is operator

$T_2 : R \longrightarrow R^2$ $x \in R^2$, $x_1, x_2 \in R$

$T_2(x_1) = (x_1, 2x_1)$

Definition Linear operator

Let $(X, +_x, \cdot_x, K)$ & $(Y, +_Y, \cdot_Y, K)$ be two vectors spaces over the same field K, and T is an operator from X into Y. T is said to be linear operator when

① $T(x_1 +_x x_2) = T(x_1) +_Y T(x_2)$

② $T(\alpha \cdot_x x) = \alpha \cdot_Y T(x)$

$\forall x_1, x_2, x \in X$, $\alpha \in K$

we use

$D(T)$ to denoted the domain of T

$R(T)$ to denoted the range of T

for this definition $D(T) = X$, $R(T) \subseteq Y$

we can re-write the two condition as

$T(\alpha x_1 + \beta x_2) = \alpha \cdot T(x_1) + \beta T(x_2)$

Remark we must know that the operation and scalar multiplication inside the operator it is belong to domain vector space and the outside operator belonge to codomain.

Definition **Null space**

Let T be an operator for vector space X into vector space Y. The null space of T is the set of all vectors $x \in X$ such that $T(x) = 0_Y$ and denoted it by $N(T)$

$N(T) = \{ x \in X , T(x) = 0 \}$

$N(T) \subseteq X$

Definition **Range**

Let $T : X \to Y$ be an operator from vector space X into Y, Then the range of T denoted by $R(T)$ define as

$R(T) = \{ T(x) : x \in X \}$ we see that

$R(T) \subseteq Y$

Example ① **The identity operator.**

Let X be a vector space, The operator from X into X is called identity operator if the image of every element in X is the same element in X and denoted it by I for vector space X we denoted it by I_X.

$I_X : X \to X$, $I_X(x) = x$ for all $x \in X$

I_x is linear operator, since

for $x, x_1, x_2 \in X$, $\alpha \in K$

$I_x(x_1 + x_2) = x_1 + x_2$

$= I_x(x_1) + I_x(x_2)$

$I_x(\alpha x_1) = \alpha x_1$

$= \alpha I(x_1)$

$\therefore I_x$ is a linear operator.

The nullspace of I_x

$N(I_x) = \{0\}$, since $I_x(0) = 0$

is the only element $0 \in X$ have image 0

$R(I_x) = X$

since $\forall x \in X$, $\exists x \in X$ such that $I_x(x) = x$

for R^2 the $I : R^2 \longrightarrow R^2$

$x \in R^2$

$I(x) = I(x_1, x_2) = (x_1, x_2)$

for $M_{2\times 2}(R)$

$I : M_{2\times 2}(R) \longrightarrow M_{2\times 2}(R)$

$I\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Example 2 Zero operator

Let X, Y be two vector spaces over the same field, The operator from X into Y is called zero operator, if the image of every element in X is equal to zero element in Y

$O : X \longrightarrow Y \quad , O(x) = \theta \quad \forall x \in X$

O is linear operator

Let $x_1, x_2 \in X$

$O(x_1 + x_2) = \theta = \theta + \theta$

$= O(x_1) + O(x_2)$

For $x \in X$, $\alpha \in K$

$O(\alpha x) = \theta = \alpha . \theta$

$= \alpha . O(x)$

The nullspace $N(O) = X$

$R(O) = \{\theta\}$

for $O : R \rightarrow R^2$

$O(x) = (0,0)$

for $O : R^2 \longrightarrow R^4$

$O(x) = (0,0,0,0)$

Note

Let T be an operator $T: D(T) \longrightarrow Y$

* The dimension of $R(T)$, denoted by $\dim(R(T))$, is also called the rank of T and denoted by $\text{rank}(T)$.

* The dimension of $N(T)$, $\dim(N(T))$ is also called nullity of T, denoted by $\text{nullity}(T)$

we can see that

$$\dim(R(T)) + \dim(N(T)) = \dim(D(T))$$

Example 3 differentation operators

Let X be the vector space of all polynomials on $[a,b]$. we may define a linear operator T on X by setting $T: X \longrightarrow X$

$$T(f(t))(t) = f'(t) \quad \text{for all } f \in X$$

where the prime denotes to the differentiation with respect to t.

T is linear operator.

(8)

Let $f_1, f_2 \in X$

$$T((f_1+f_2)(t)) = (f_1+f_2)'(t)$$

$$= (f_1' + f_2')(t)$$

$$= f_1'(t) + f_2'(t)$$

$$= (T(f_1))(t) + (T(f_2))(t)$$

$$= \left[T(f_1) + T(f_2)\right)(t) \quad \forall t \in [a,b]$$

$$\therefore T(f_1+f_2) = T(f_1) + T(f_2)$$

Let $f \in X$, $\alpha \in R$

$$T(\alpha f)(t) = (\alpha f)'(t)$$

$$= \alpha . f'(t)$$

$$= \alpha \cdot T(f)(t) \quad \forall t \in [a,b]$$

$$\therefore T(\alpha f) = \alpha . T(f)$$

$$N(T) = \{ f \in X : f(t) = c , c \in R \}$$

$$R(T) = X$$

Example Integration operator

A linear operator $T: C[a,b] \longrightarrow C[a,b]$ define by

$$T(f)(t) = \int_a^t f(x)\,dx \qquad t \in [a,b]$$

is linear operator.

Let $f_1, f_2 \in C[a,b]$.

$$T(f_1+f_2)(t) = \int_a^t (f_1+f_2)(x)\,dx$$

$$= \int_a^t (f_1(x)+f_2(x))\,dx$$

$$= \int_a^t f_1(x)\,dx + \int_a^t f_2(x)\,dx$$

$$= T(f_1)(t) + T_2(f_2)(t)$$

$$= (T(f_1) + T_2(f_2))(t) \qquad \forall t \in [a,b]$$

$$\therefore T(f_1+f_2) = T(f_1) + T(f_2)$$

Let $f \in C[a,b]$, $\alpha \in R$

$$T(\alpha f)(t) = \int_a^t (\alpha f)(x)\,dx$$

$$= \int_a^t \alpha f(x)\,dx$$

$$= \alpha \int_a^t f(x)\,dx$$

$$= (\alpha T(f))(t) \qquad \forall t \in [a,b]$$

$$\therefore T(\alpha f) = \alpha T(f).$$

(12) Example multiplication by t

Another linear operator from $C[a,b]$ into $C[a,b]$

$T: C[a,b] \to C[a,b]$ defined by

$T(f)(t) = t \cdot f(t) \qquad t \in [a,b]$

$\qquad\qquad\qquad\qquad f \in C[a,b]$

Let $f_1, f_2 \in C[a,b]$.

$$\begin{aligned}
T(f_1+f_2)(t) &= t \cdot (f_1+f_2)(t) \\
&= t(f_1(t)+f_2(t)) \\
&= t \cdot f_1(t) + t\, f_2(t) \\
&= T(f_1)(t) + T(f_2)(t) \\
&= (T(f_1)+T(f_2))(t) \qquad \forall t \in [a,b]
\end{aligned}$$

$$T(f_1+f_2) = T(f_1) + T(f_2)$$

Let $f \in C[a,b]$, $\alpha \in \mathbb{R}$

$$\begin{aligned}
T(\alpha f)(t) &= t \cdot (\alpha f)(t) \\
&= t \cdot (\alpha f(t)) \\
&= \alpha \cdot t \cdot f(t) \\
&= \alpha \cdot T(f)(t) \\
&= (\alpha \cdot T(f))(t) \qquad \forall t \in [a,b]
\end{aligned}$$

$\therefore T(\alpha f) = \alpha \cdot T(f)$

So T is linear operator.

Example $T: R^3 \to R$ for $x \in R^3$

$x = (x_1, x_2, x_3)$ $x_1, x_2, x_3 \in R$

$T(x) = T((x_1, x_2, x_3)) = x_1 + x_2 + x_3$

T is linear operator

Let $x, y \in X$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

$$T(x+y) = T((x_1, x_2, x_3) + (y_1, y_2, y_3))$$
$$= T(x_1+y_1, x_2+y_2, x_3+y_3)$$
$$= x_1 + y_1 + x_2 + y_2 + x_3 + y_3$$
$$= x_1 + x_2 + x_3 + y_1 + y_2 + y_3$$
$$= T(x) + T(y)$$

Let $\alpha \in R$

$$T(\alpha x) = T(\alpha(x_1, x_2, x_3))$$
$$= T(\alpha x_1, \alpha x_2, \alpha x_3)$$
$$= \alpha x_1 + \alpha x_2 + \alpha x_3$$
$$= \alpha(x_1 + x_2 + x_3)$$
$$= \alpha \cdot T(x)$$

$N(T) = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 , x_1, x_2, x_3 \in R\}$

$R(T) = R$.

Example Matrices

Let $A_{m \times n}$ be a Matrix of real number with dimension $m \times n$, the operator T

$T: R^n \longrightarrow R^m$ defenied by

$$T(X_{n\times 1}) = A_{m\times n}\, \bar{X}_{n\times 1} = Y_{m\times 1}$$

$x \in R^n$, $Y \in R^m$

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Then T is a linear operator.

Example $T: R^2 \longrightarrow R^2$ Prove that

$T(x) = T(x_1, x_2) = (x_2, x_1)$ $\quad x \in R^2$, $x_1, x_2 \in R$

T is linear

Let $x, y \in R^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ $\quad x_1, x_2, y_1, y_2 \in R$

$T(x+y) = T((x_1, x_2) + (y_1, y_2)) = T(x_1+y_1, x_2+y_2)$

$= (x_2+y_2, x_1+y_1) = (x_2, x_1) + (y_2, y_1)$

$= T(x) + T(y)$

$T(\alpha x) = T(\alpha(x_1, x_2)) = T(\alpha x_1, \alpha x_2)$

$= (\alpha x_2, \alpha x_1) = \alpha(x_2, x_1)$

$= \alpha T(x)$.

T is linear operator

$N(T) = \{(0,0)\}$

$R(T) = R^2$

Theorem (Range and null space)

Let T be a linear operator Then:
$T: X \longrightarrow Y$, X, Y are vector spaces

1. The range $R(T)$ is a vector subspace of Y
2. if $\dim X = n < \infty$, then $\dim R(T) \leq n$
3. The null space $N(T)$ is a vector subspace of X

proof. 1. Let $y_1, y_2 \in R(T)$ and $\alpha, \beta \in K$
we to show that $\alpha y_1 + \beta y_2 \in R(T)$
since $y_1, y_2 \in R(T)$, $\exists\, x_1, x_2 \in X$, such that

$T(x_1) = y_1$, $T(x_2) = y_2$

$$T(\alpha x_1 + \alpha x_2) = \alpha T(x_1) + \beta T(x_2) \quad \text{since } T \text{ is linear}$$
$$= \alpha y_1 + \beta y_2$$

hence $\alpha y_1 + \beta y_2 \in R(T)$, since $y_1, y_2 \in R(T)$ were arbiturary and so the scalars, this prove that $R(T)$ is vector subspace of Y.

2. we choose $n+1$ elements $y_1, y_2 \cdots y_n, y_{n+1}$ of $R(T)$ an arbitrary element, we will explain that this set of element are linearly dependent
since $\quad y_i \in R(T) \quad 1 \leq i \leq n+1$ then there exi.
$\quad x_i \in X \quad 1 \leq i \leq n+1$ such that

$$T(x_i) = y_i$$

(4) Since the $\dim(X) = n$, then the set $\{x_i : 1 \le i \le n+1\}$ is linearly dependent, that is mean

$$\alpha x_1 + \alpha_2 x_2 + \dots \alpha_n x_n + \alpha_{n+1} x_{n+1} = 0$$

there exist i, $1 \le i \le n+1$ such that $\alpha_i \ne 0$

now

$$T(\alpha_1 x_1 + \dots \alpha_n x_n + \alpha_{n+1} x_{n+1}) = T(0) = 0$$

$$T(\alpha_1 x_1 + \dots \alpha_n x_n + \alpha_{n+1} x_{n+1}) = \alpha_1 T(x_1) + \dots \alpha_{n+1} T(x_{n+1})$$

$$= \alpha_1 y_1 + \dots \alpha_n y_n + \alpha_{n+1} y_{n+1}$$

$$= 0$$

The linear combination of $n+1$ element are equal zero but not all α_i are equal to zero, then this mean linear combination are linearly dependent, we conclude that $R(T)$ has no linearly independent subset of $n+1$ or more elements. That is mean

$$\dim(R(T)) \le n.$$

(3) Let $x_1, x_2 \in N(T)$, Then $T(x_1) = 0$, $T(x_2) = 0$

Let $\alpha, \beta \in K$, we want to show that $\alpha_1 x_1 + \alpha_2 x_2 \in N(T)$

$$T(\alpha_1 x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$$

$$= \alpha \cdot 0 + \beta \cdot 0$$

$$= 0$$

This is show that $\alpha x_1 + \beta x_2 \in N(T)$

is a subspace of X.