Lemma: every subspace M of vectorspace X over filed K is a convex set.

proof: Let M be a subspace of X then $\alpha x + \beta y \in M$, $\forall \alpha, \beta \in K$ and $x, y \in M$

Let $0 \leq \alpha \leq 1$ and $\beta = (1-\alpha)$ then

$$\alpha x + (1-\alpha) y \in M \quad , \forall \alpha \in R \quad \{0 \leq \alpha \leq 1\}$$

$\therefore$ M is convex set.

Lemma: Let M_1, M_2 be two convex subset of vectorspace X, then

① $M_1 + M_2$ is convex set

~~② $M_1 \cap M_2$ is convex set~~

② $M_1 \cap M_2$ is convex set

proof: Let $\cancel{x}, y \in M_1 + M_2$, and $\alpha \in K$

$\because$ $x, y \in M_1 + M_2$ $\qquad 0 \leq \alpha \leq 1$

$\exists$ x_1, y_1 $\in M_1$ and $x_2, y_2 \in M_2$

~~so $x = a_1 + a_2$~~

so $x = x_1 + x_2$

$y = y_1 + y_2$

$\because$ $x_1, y_1 \in M_1$ which is convex, so

$\alpha x_1 + (1-\alpha) y_1 \in M_1$

$\because x_2, y_2 \in M_2$ which is convex so

$\alpha x_2 + (1-\alpha) y_2 \in M_2$

now $\in M_1 + M_2$

$\alpha x_1 + (1-\alpha) y_1 + \alpha x_2 + (1-\alpha y_2)$

$= \alpha x_1 + \alpha x_2 + (1-\alpha) y_1 + (1-\alpha) y_2$

$= \alpha (x_1 + x_2) + (1-\alpha)(y_1 + y_2)$

$= \alpha x + (1-\alpha) y \in M_1 + M_2$

$\therefore M_1 + M_2$ is convex

~~Let $x, y \in M_1 \cap M_2$~~

~~So x~~

(2) Let $x, y \in M_1 \cap M_2$

$\because x, y \in M_1$, then for $\alpha \in \mathbb{R}$ $(0 \le \alpha \le 1)$

$\alpha x + (1-\alpha) y \in M_1$

also $x, y \in M_2$, then

$\alpha x + (1-\alpha) y \in M_2$

So $\alpha x + (1-\alpha) y \in M_1 \cap M_2$

$\therefore M_1 \cap M_2$ is convex set.

Example Let $X = R^2$ be a vector space

Then $M_1 = \{ (x, 0), x \in R \}$

$M_2 = \{ (0, x), x \in R \}$

$M_3 = \{ (z, w) : z, w \in R, z \geq 0, w \geq 0 \}$

M_1, M_2, M_3 are convex set.

Solution in previous example we show that M_1, M_2 are convex, for M_3

Let $x, y \in M_3$, $x = (x_1, x_2)$

$y = (y_1, y_2)$

$x_1, x_2, y_1, y_2 \in R$

for $\alpha \in R$, $0 \leq \alpha \leq 1$ and $x_1, x_2, y_1, y_2 \geq 0$

$$\alpha x + (1-\alpha) y = \alpha (x_1, x_2) + (1-\alpha)(y_1, y_2)$$

$$= (\alpha x_1, \alpha x_2) + ((1-\alpha) y_1, (1-\alpha) y_2)$$

$$= (\alpha x_1 + (1-\alpha) y_1, \alpha x_2 + (1-\alpha) y_2)$$

$\because x_1 \geq 0$ and $0 \leq \alpha \leq 1$ so $\alpha x_1 \geq 0$ — ①

and $y_1 \geq 0$ $0 \leq \alpha \leq 1$ so $1-\alpha \geq 0$

$\therefore (1-\alpha) y_1 \geq 0$ — ②

from ① & ② we get $\alpha x_1 + (1-\alpha) y_1 \geq 0$

Similarly for $\alpha x_2 + (1-\alpha) y_2 \geq 0$

$\therefore \alpha x + (1-\alpha) y \in M_3$

$\therefore M_3$ is convex

(14)

$M_1 + M_2 = \{ (x, y) / x, y \in R \} = R^2$

is convex

$M_1 + M_3 = \{ (x, y) / x, y \in R \text{ and } y \geq 0 \}$

is convex

$M_2 + M_3$ ~~$= \{(x,$~~ Home work ?

$M_1 \cap M_2 = \{ (0, 0) \}$

is convex

$M_1 \cap M_3 = \{ (x, 0) / x \in R, x \geq 0 \}$

is convex

$M_2 \cap M_3 = \{ (0, y) / y \in R, y \geq 0 \}$

is convex.

---

we see that M_3 is convex set but it is not sub space of R^3

because $-2(x, y) \notin M_3$

$-2(x, y) = (-2x, -2y)$

$\because x \geq 0$

$-2x \leq 0$

also $y \geq 0$

$-2y \leq 0$

so $(-2x, -2y) \notin M_3$

M_3 is not subspace $\alpha M \notin M \quad \forall \alpha \in R$

not closed under scalar multiplication

Theorem ~~...~~ (Minimizing Vector)

Let X be a pre Hilbert space and $M \neq \phi$ be a convex subset which is complete. Then for every given $x \in X$, there exists a unique $y \in M$ such that

$$\delta = \inf_{z \in M} \{\|x-z\|\} = \|x-y\|$$

Example In previous example $X = R^2$, $M_1 \subseteq R^2$. Let $x = (0, 2)$, with $z \in R^2$, $z = (z_1, z_2)$, $\|z\| = |z_1| + |z_2|$

~~$\delta = \inf_{z \in M_1} \{\|x-z\|\}$~~ find distance between x and M_1

Solution

$$\delta = \inf_{z \in M_1} \{\|x-z\|\}$$

$$= \inf_{z_1 \in M_1} \{\|(0,2)-(z_1,0)\|\}$$

$$= \inf_{z_1 \in M_1} \{\|(-z_1, 2)\|\}$$

$$= \inf_{z_1 \in M_1} \{|-z_1| + |2|\}$$

$$= \inf_{z_1 \in M_1} \{|z_1| + 2\}$$

$\delta = 2$ is the distance between x and M_1

by The theorem $\exists\, y \in M_1$ such that $\delta = \|x-y\|$

we get $y = (0,0)$, $\delta = \|x-y\| = \|(0,2)-(0,0)\|$

$= \|(0,2)\| = |0| + |2|$

$= 2$

~~$\delta = \| x$~~

pre Hilbert

Lemma

Let M be a complete subspace X and $x \in X$ is fixed. Then $z = x - y$ is orthogonal to M when $y \in M$ is minimizing vector.

Example Let $X = R^2$, $M_1 = \{(x, 0), x \in R\}$ with $x = (0, 2)$ is complete subspace of R^2

we find $\delta = 2$ and $y = (0, 0)$

so $z = x - y = (0, 2) - (0, 0) = (0, 2)$ is orthogonal to M

for any $w \in M_1$, $w = (w_1, 0)$

$$\langle w / x \rangle = \langle (w_1, 0) / (0, 2) \rangle$$

$$= w_1 \cdot \bar{0} + 0 \cdot \bar{2}$$

$$= 0 + 0$$

$$= 0 \qquad \forall w \in M_1$$

i.e. $x \perp M_1$

~~also~~

definition direct sum مجموع مباشر

A vector space X is said to be the direct sum of two subspaces Y and Z of X written $X = Y \oplus Z$

if ① $Y \cap Z = \phi$ الانفصال

② $Y + Z = X$

in another word for each $x \in X$ has unique representation $x = y + z$ $y \in Y$, $z \in Z$

Then Z is called algebraic complement of Y in X and vice versa.

Example In R^2, we have

$M_1 = \{(x, 0), x \in R\}$

$M_2 = \{(0, x), x \in R\}$

$R^2 = M_1 \oplus M_2$ التآلف

definition orthogonal complement. المتمم العمودي (annihilator)

Let M be a subspace of pre Hilbert space X the ~~the~~ orthogonal complement of M is

$$M^{\perp} = \{x \in X / x \perp M\}$$

which is the set of all vectors orthogonal to M

Example Let $\bar{X} = \mathbb{C}^2$, with inner product

$x = (x_1, x_2)$ where $x, y \in \mathbb{C}^2$
$y = (y_1, y_2)$

$\langle x / y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_4$

so $M_1 = \{ (x_1, 0), x_1 \in \mathbb{C} \}$

$\cancel{M_2 = \{ (0, x_2), x_2 \in \mathbb{C} \}}$

$M_1^{\perp} = \{ z \in \bar{X} / z \perp M \}$ $\bar{X}$

Let $x \in M$, so

$\langle x / z \rangle = \langle (x_1, 0), (z_1, z_2) \rangle$

$= x_1 \bar{z}_1 + 0 \cdot \bar{z}_2$

$= x_1 \bar{z}_1 \quad = 0$

if and only if $\bar{z}_1 = 0 \Rightarrow z_1 = 0$

so $M^{\perp} = \{ (0, z_2) / z_2 \in \mathbb{Z} \}$

Theorem Direct sum

Let M be any closed subspace of Hilbert space $\bar{X}$. then

$$\bar{X} = M \oplus M^{\perp}$$

proof. since X is Hilbert space, it is complete, $\because$ M is closed subspace so M is also complete. By

" Every closed subspace of complete space is complete "

$\because$ M is subspace so it is convex set by using minimizing vector for any $x \in M$, $\exists$ unique $y \in M$ and by using orthogonal lemma $z = x - y$ we have $z \perp M$

that is mean $z \in M^{\perp}$

$x = y + z$, $y \in M$, $z \in M^{\perp}$

that is mean $X = M + M^{\perp}$

now let ~~x~~ $w \in M \cap M^{\perp} \longrightarrow w \perp w$

$\longrightarrow \langle w / w \rangle = 0$

$\longrightarrow w = 0$

~~so X =~~ $= M \cap M^{\perp} = \{0\}$

so $X = M + M^{\perp}$

~~Lemma Let X be~~

In previous Theorem $x = y + z$

and $y \in M$ المسقط العامودي

y is called the orthogonal projection of x on M

$P : X \longrightarrow M$

$x \longrightarrow P(x) = y$

المسقط مؤثر

this is the projection operator

المسقط العامودي المسقط

or orthognal projection المسقط

$P : X \longrightarrow M$

$X = M + M^{\perp}$

$P(M) = M$

$P(M^{\perp}) = 0$

P is in an idempotent oprator

$P^2 = P$

$$P^2(x) = P(\underbrace{P(x)}_{\in M}) = P(x)$$

~~$P(M)$ is~~

$P : M \longrightarrow M$ it will be identity operator.

P is linear bounded operator

prove that (H.W.)

Example $X = \mathbb{C}^2$

$M = \{ (x, 0) / x \in \mathbb{C} \}$

$P : X \longrightarrow M$ with $\|(x,y)\| = |x| + |y|$

$P(x,y) = (x, 0)$

is an projection operator

we see that

$P(M) = M$, since $P(x,0) = (x,0)$ $\forall (x,0) \in M$

$M^{\perp} = \{ (0, x) , x \in \mathbb{C} \}$

$P(M^{\perp}) = 0$, $P(0, x) = (0,0)$ $\forall (0,x) \in M^{\perp}$

$P^2(x,y) = P(P(x,y)) = P(x,0) = (x,0)$

$P(x,y) = (x,0)$

$\therefore P^2(x,y) = P(x,y)$

Let $w = (x_1, y_1)$

$v = (x_2, y_2)$

$P(w+v) = P((x_1,y_1) + (x_2,y_2))$

$= P(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2, 0)$

$= (x_1, 0) + (x_2, 0)$

$= P(x_1, y_1) + P(x_2, y_2)$

$= P(w) + P(v)$

$P(\alpha w) = P(\alpha(x_1, y_1)) = P(\alpha x_1, \alpha y_1)$

$= (\alpha x_1, 0) = \alpha(x_1, 0) = \alpha P(x_1, y_1) = \alpha P(w)$

$\therefore P$ is linear operator

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P is bounded

let $w = (x,y) \in \mathbb{C}^2$

$$\|P(w)\| = \|P(x,y)\|$$
$$= \|(x,0)\|$$
$$= |x|$$
$$\leq |x| + |y|$$
$$= \|w\|$$

$\therefore \|P(w)\| \leq \|w\| \quad \forall w \in \mathbb{C}^2$

$\therefore$ P is bounded operator.

Lemma Let X be a pre Hilbert space then

(1) ~~$M^\perp$~~ and M is a subspace of X, then

(1) $M^\perp$ is a sub space of X

(2) $M^\perp$ is closed

(3) $M \subseteq (M^\perp)^\perp$

proof (1) Let $x, y \in M^\perp$, then for all $v \in M$

$\langle x / v \rangle = 0$
$\langle y / v \rangle = 0$

Let $\alpha, \beta \in K$

$$\langle \alpha x + \beta y / v \rangle = \alpha \langle x / v \rangle + \beta \langle y / v \rangle$$
$$= \alpha \cdot 0 + \beta \cdot 0$$
$$= 0$$

$\therefore \alpha x + \beta y \in M^\perp$

proof ② Let ~~$\{x_n\}$~~ $x_n \in M^\perp$ be sequence

and $x_n \to x$ we to show that $x \in M^\perp$

then $M^\perp$ is closed

$\because x_n \in M^\perp$ then for all $v \in M$ we have

$\langle x_n / v \rangle = 0$

$\because x_n \longrightarrow x$, So by using the continuous of inner product.

$\langle x_n / v \rangle \longrightarrow \langle x / v \rangle$

that is mean

$\langle x_n / v \rangle = 0 \longrightarrow 0 = \langle x / v \rangle$

so $\langle x / v \rangle = 0 \quad \forall v \in M$

so $x \in M^\perp$ $\quad \therefore M^\perp$ is closed.

proof ③ Let $x \in M$, then $\forall w \in M^\perp$

$\langle x / w \rangle = 0 \quad \forall w \in M^\perp$

we see that $x \perp M^\perp \Rightarrow x \in M^{\perp\perp}$

Since $M^\perp = \{ x \in X / x \perp M \}$

$M^{\perp\perp} = \{ x \in X / x \perp M^\perp \}$

$\therefore M \subseteq M^{\perp\perp}$

Lemma

Let M be closed subspace of ~~X~~ Hilbert space X, then $M = M^{\perp\perp}$

Lemma

For any subset $M \neq \phi$ of a Hilbert space X, the (span of M) is dense in X if and only if $M^{\perp} = \{0\}$.

Ex For $X = \mathbb{R}^2$, $M = \{(x,0), x \in \mathbb{R}\}$

then $M^{\perp} = \{(0,x), x \in \mathbb{C}\}$

which is also subspace of X

~~$M^{\perp}$ is closed~~

$x, y \in M^{\perp}$. $x = (0, x_2)$

$y = (0, y_2)$

~~$\alpha x + \beta y = \alpha(0, x_2) +$~~

$x + y = (0, x_2) + (0, y_2)$

$= (0, x_2 + y_2)$ $\because x_2 + y_2 \in \mathbb{C}$

$\therefore (0, x_2 + y_2) \in M^{\perp}$

Let $\alpha \in \mathbb{C}$

$\alpha x = \alpha(0, x_2) = (0, \underbrace{\alpha x_2}_{\in \mathbb{C}}) \in M^{\perp}$

$\therefore M^{\perp}$ is a subspace of X

$M^{\perp}$ is closed

so $M = M^{\perp\perp}$ since M is closed and $\mathbb{C}^2$ is Hilbert space

by using Lemma
" M be closed subspace of Hilbert space X then $M = M^{\perp\perp}$ "

~~ex~~ Definition orthonormal sets

Let M be a subset of pre Hilbert space X

we say that M is an orthonormal set

① $x \perp y$ means $\langle x / y \rangle = 0 \quad \forall x, y \in M$

② $\|x\| = 1 \quad \forall x \in M$

Ex in $X = \mathbb{C}^2$, then $M = \{(1,0), (0,1)\}$ is orthonormal set

$(1,0) \perp (0,1)$

since $\langle (1,0) / (0,1) \rangle = 1 \cdot \bar{0} + 0 \cdot \bar{1} = 0$

$\|(1,0)\| = \sqrt{|1|^2 + |0|^2} = 1$

$\|(0,1)\| = \sqrt{|0|^2 + |1|^2} = 1$

Ex in R^3, if $M = \{(1,0,0), (0,1,0), (0,0,1)\}$ is orthonormal set.

H.W.

عمليات كرام شميت

~~Theorem~~ Gram - Schmidt process

If $x_1, x_2, \ldots, x_n, \ldots$ is a sequence of linearly independent vectors in pre-Hilbert space X. Then there exists an orthonormal sequence $y_1, y_2, y_3, \ldots, y_n, \ldots$

The process

(1) $y_1 = \frac{1}{\|x_1\|} x_1$

(2) $v_2 = x_2 - \langle x_2 / y_1 \rangle y_1$

$y_2 = \frac{1}{\|v_2\|} v_2$

(3) $v_3 = x_3 - \langle x_3 / y_1 \rangle y_1 - \langle x_3 / y_2 \rangle y_2$

$y_3 = \frac{1}{\|v_3\|} v_3$

(n) $v_n = x_n - \sum_{k=1}^{n-1} \langle x_n / y_k \rangle y_k$

$y_n = \frac{1}{\|v_n\|} v_n$

Example Let $X = \mathbb{C}^2$ and if $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2}$ $x, y \in X$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$x_1 = (1+i, i)$

$x_2 = (3, 1-i)$

Show that x_1, x_2 are linearly independent.

find the orthonormal set of x_1, x_2

Solution let $\alpha_1, \alpha_2 \in \mathbb{C}$, then

$\alpha_1 x_1 + \alpha_2 x_2 = (0, 0)$

$\alpha_1(1+i, i) + \alpha_2(3, 1-i) = (0, 0)$

$(\alpha_1 + \alpha_1 i, \alpha_1 i) + (3\alpha_2, \alpha_2 - \alpha_2 i) = (0, 0)$

$(\alpha_1 + \alpha_1 i + 3\alpha_2, \alpha_1 i + \alpha_2 - \alpha_2 i) = (0, 0)$

$\alpha_1 + 3\alpha_2 + \alpha_1 i = 0 \longrightarrow \alpha_2 = 0$

$\alpha_2 + (\alpha_1 - \alpha_2) i = 0$ $\quad \alpha_1 + 3\alpha_2 = 0$

$\therefore \alpha_1 = \alpha_2 = 0$ $\quad \alpha_1 = 0$

So x_1, x_2 are linearly independent.

To find the orthonormal set of x_1, x_2 we use Gram - Schmidt process

① $y_1 = \frac{1}{\|x_1\|} x_1$

$\|x_1\|^2 = \langle x_1, x_1\rangle = \langle (1+i, i), (1+i, i)\rangle$

$= 1+i \, \overline{1+i} + i \, \overline{i}$

$= 1+1+1 = 3$

$\|x_1\| = \sqrt{3}$

$$y_1 = \frac{1}{\|x_1\|} x_1 = \frac{1}{\sqrt{3}}(1+i, i)$$

$$= \left(\frac{1}{\sqrt{3}} + \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}\right)$$

② $$v_2 = x_2 - \langle x_2 / y_1 \rangle y_1$$

$$\text{so } \langle x_2 / y_1 \rangle = \left\langle (3, 1-i) / \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} i, \frac{1}{\sqrt{3}} i\right) \right\rangle$$

$$= 3 \cdot \overline{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} i} + (1-i)\left(\overline{\frac{1}{\sqrt{3}} i}\right)$$

$$= 3 \cdot \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} i\right) + (1-i)\left(\frac{-1}{\sqrt{3}} i\right)$$

$$= \sqrt{3} - \sqrt{3} i + \frac{-1}{\sqrt{3}} i - \frac{1}{\sqrt{3}}$$

$$= \sqrt{3} - \frac{1}{\sqrt{3}} - \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) i$$

$$\langle x_2 / y_1 \rangle y_1 = \left(\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) i\right)\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} i, \frac{i}{\sqrt{3}}\right)$$

$$= \left(\left[\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) - \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\right] + \left[\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) i\right], \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)\frac{i}{\sqrt{3}} + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) i \frac{i}{\sqrt{3}}\right)$$

اعدل

تأتي

$$= \left(\cancel{1} - \frac{1}{3} \cancel{-1} - \frac{1}{3}\right) + \left(1 + \cancel{\frac{1}{3}} + 1 - \cancel{\frac{1}{3}}\right) i, \left(1 - \frac{1}{3}\right) i + \left(1 + \frac{1}{3}\right)$$

$$= \left(-\frac{2}{3} + 2i, \cancel{\frac{1}{3}} -\frac{4}{3} + \frac{2}{3} i\right)$$

$$v_2 = (3, 1-i) - \left(-\frac{2}{3} + 2i, -\frac{4}{3} + \frac{2}{3} i\right)$$

$$= \left(3 + \frac{2}{3} - 2i, 1 - i + \frac{4}{3} - \frac{2}{3} i\right)$$

$$= \left(\frac{11}{3} - 2i, \frac{7}{3} - \frac{5}{3} i\right)$$

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$$y_2 = \frac{1}{\|v_2\|} v_2$$

$$\|v_2\|^2 = \langle v_2 / v_2 \rangle = \left(\frac{11}{3} - 2i, \frac{7}{3} - \frac{5}{3}i\right) / \left(\frac{11}{3} - 2i, \frac{7}{3} - \frac{5}{3}i\right)$$

$$= \left(\frac{11}{3} - 2i\right) \cdot \overline{\left(\frac{11}{3} - 2i\right)} + \left(\frac{7}{3} - \frac{5}{3}i\right) \cdot \overline{\left(\frac{7}{3} - \frac{5}{3}i\right)}$$

$$= \left(\frac{11}{3}\right)^2 + 2^2 + \left(\frac{7}{3}\right)^2 + \left(\frac{5}{3}\right)^2$$

$$= \frac{121}{9} + 4 + \frac{49}{9} + \frac{25}{9}$$

$$= \frac{195}{9} + \frac{36}{9} = \frac{231}{9}$$

$$\begin{array}{r} {}^{1}\\ 195 \\ 36 \\ \hline 231 \end{array}$$

$$\|v_2\| = \frac{\sqrt{231}}{3}$$

$$y_2 = \frac{\sqrt{231}}{3}\left(\frac{11}{3} - 2i, \frac{7}{3} - \frac{5}{3}i\right)$$

$$= \left(\frac{11\sqrt{231}}{9} - \frac{2\sqrt{231}}{3}, \frac{7\sqrt{231}}{9} - \frac{5\sqrt{231}}{9}i\right)$$