Green's Theorem
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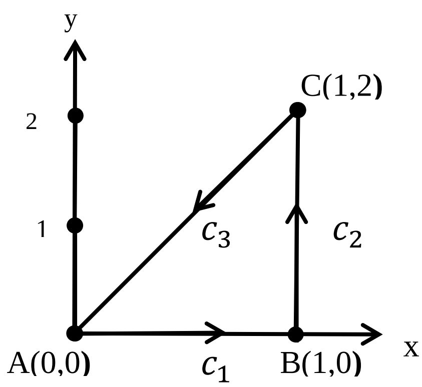
Theorem: Let R be a closed and bounded region of the xy −plane. Let the boundary of R contains a simple closed curve C . Let $f(x,y)$ and $g(x,y)$ be functions of two variable which are continuous and have continuous first partial derivatives in R. Then

$$\oint_C [f(x,y)dx + g(x,y)dy] = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

Where $\oint_C$ is line integral counterclockwise direction.

Example 1: Use green's theorem to evaluate $\oint_C x^2ydx + x\,dy$ along triangular path $A(0,0), B(1,0), C(1,2)$ to A.

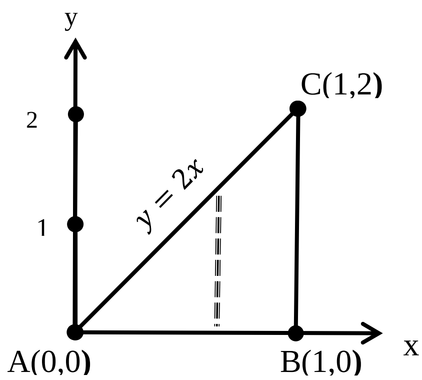
Solution: c_3: $\text{y} = 2\text{x}, dy = 2\,dx$



$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} \implies \frac{x-1}{0-1} = \frac{y-2}{0-2} \implies \text{y} = 2\text{x}$$

$$f(x,y) = x^2y\,, \qquad g(x,y) = x$$

$$\frac{\partial g}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = x^2$$



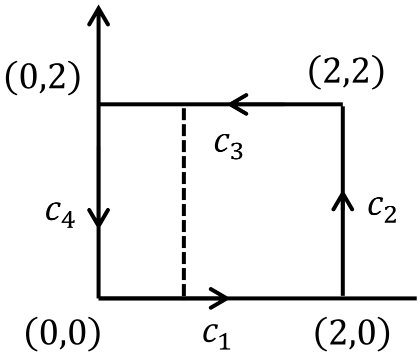
$$\oint_C x^2ydx + x\,dy = \iint_R [1 - x^2]\,dA$$

$$= \int_0^1\int_0^{2X} (1-x^2)\boldsymbol{dydx} = \int_0^1 \boldsymbol{y - x^2y}\Big|_0^{2x} \boldsymbol{dx}$$

$$= \int_0^1 (\boldsymbol{2x - 2x^3})\boldsymbol{dx} = \boldsymbol{x^2 - \frac{x^4}{2}}\Big|_0^1 = \boldsymbol{1 - \frac{1}{2} = \frac{1}{2}}$$

Example 2: Evaluate $\oint_C \; y^2 dx + x^2\, dy$ along rectangular path from $A(0,0), B(2,0), C(2,2), D(0,2)$ to A.

Solution:



$f(x,y) = y^2$, $\qquad g(x,y) = x^2$

$$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = \; 2y$$

$$\oint_C \; y^2 dx + x^2\, dy = \iint_R [2x - 2y]\, dA$$

$$= \int_{\mathbf{0}}^{\mathbf{2}} \int_0^2 (2x - 2y) dy dx = \int_{\mathbf{0}}^{\mathbf{2}} \mathbf{2xy - y^2}\Big|_0^{\mathbf{2}} dx = \int_{\mathbf{0}}^{\mathbf{2}} (\mathbf{4x - 4})\, dx = \; \mathbf{2x^2 - 4x}\Big|_{\mathbf{0}}^{\mathbf{2}} = \mathbf{0}$$

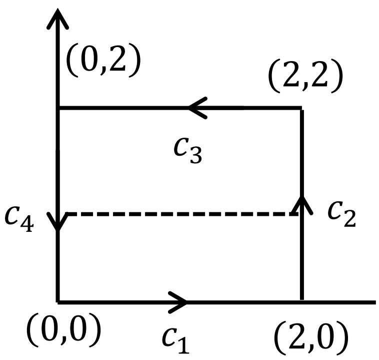
Second method (line integral)

c_1: $y = 0 \quad \Rightarrow \quad dy = 0$
c_2: $x = 2 \quad \Rightarrow \quad dx = 0$
c_3: $y = 2 \quad \Rightarrow \; dy = 0$
c_4: $x = 0 \quad \Rightarrow \quad dx = 0$

$$= \int_0^2 0 \; + \int_0^2 4dy \; + \int_2^0 4dx + \int_2^0 0 = 0 \; + 4y]_0^2 + 4x]_2^0 + 0 = 0$$

Example 3: Evaluate $\oint_C \; (x^3 - x^3 y)\, dx + (y^2 - 2xy)\; dy$ along rectangular path from $A(0,0), B(2,0), C(2,2), D(0,2)$ to A.

Solution:



$f(x,y) = x^3 - x^3 y$, $\qquad g(x,y) = y^2 - 2xy$

$$\frac{\partial g}{\partial x} = -2y, \quad \frac{\partial f}{\partial y} = \; -x^3$$

$$\oint_C \; (x^3 - x^3 y)\, dx + (y^2 - 2xy)\, dy = \iint_R [-2y + x^3]\, dA$$

$$= \int_{\mathbf{0}}^{\mathbf{2}} \int_0^2 (-2y + x^3) dx dy = \int_{\mathbf{0}}^{\mathbf{2}} -\mathbf{2xy} + \frac{\boldsymbol{x^4}}{\mathbf{4}}\Big|_0^{\mathbf{2}} \; dy = \int_{\mathbf{0}}^{\mathbf{2}} (-\mathbf{4y} + \mathbf{4})\, dy$$

$$= \; -\mathbf{2y^2} + \mathbf{4y}\Big|_{\mathbf{0}}^{\mathbf{2}} = \mathbf{0}$$

Theorem: Let R be a region, then the area of this region that satisfied the conditions of the Green's theorem which given by

$$A = \frac{1}{2}\oint_C xdy - ydx$$

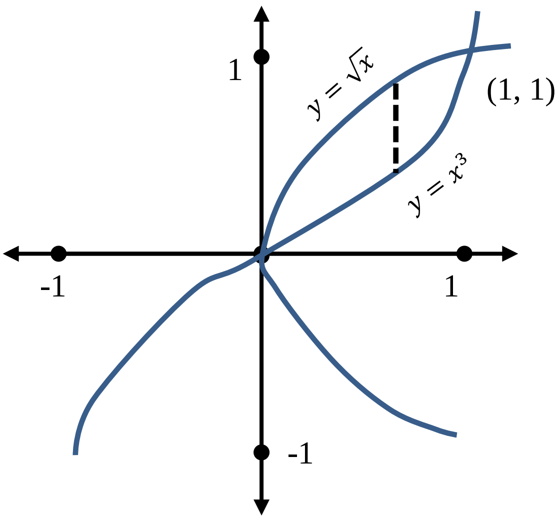
Proof:

$$f(x,y) = -y\,, \qquad g(x,y) = x \quad , \qquad \frac{\partial g}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -1$$

$$\oint_C xdy - ydx = \iint_R (1+1)\,dA = 2\iint_R dA = 2A$$

$$\oint_C xdy - ydx = 2A \quad \Rightarrow \quad A = \frac{1}{2}\oint_C xdy - ydx$$

Example 4: By using the line integral, find the area of the region bounded by two curves $y = x^3$, $y = \sqrt{x}$



Solution:

$$x^3 = x^{\frac{1}{2}} \quad \Rightarrow \quad \boldsymbol{x^6 = x} \;\Rightarrow\; \boldsymbol{x(x^5-1) = 0}$$

$$\boldsymbol{x} = 0 \quad \Rightarrow y = 0 \Rightarrow (0,0)$$

Or $\boldsymbol{x^5} = 1 \Rightarrow \boldsymbol{x} = 1 \;\Rightarrow\; y = 1 \Rightarrow (1,1)$

$$A = \frac{1}{2}\left[\int_{C_1} xdy - ydx + \int_{C_2} xdy - ydx\right]$$

$$\frac{1}{2}\left[\int_0^1 x(3x^2dx) - x^3dx + \int_1^0 x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)dx - x^{\frac{1}{2}}\,dx\right]$$

$$\frac{1}{2}\left[\int_0^1 3x^3dx - x^3dx + \int_1^0 \frac{1}{2}x^{\frac{1}{2}}dx - x^{\frac{1}{2}}\,dx\right] = \frac{1}{2}\left[\int_0^1 2x^3dx - \int_1^0 \frac{1}{2}x^{\frac{1}{2}}dx\right]$$

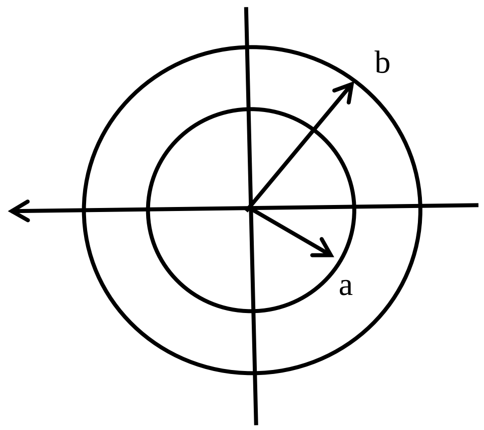
$$= \frac{1}{2}\left[\left(\frac{x^4}{2}\right)\Bigg|_0^1 - \left(\frac{1}{2}\,\frac{\boldsymbol{x^{\frac{3}{2}}}}{\boldsymbol{\frac{3}{2}}}\right)\Bigg|_1^0\right] = \frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}\right] = \frac{5}{12}$$

Other method (double integral)

$$\int_0^1\int_{x^3}^{x^{\frac{1}{2}}} dydx = \int_0^1 y\Big|_{x^3}^{x^{\frac{1}{2}}} dx = \int_0^1 x^{\frac{1}{2}} - x^3\,dx = \left.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

Example 5: Use Green's theorem to evaluate the line integral,

$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ where $c = c_1 + c_2$, $c_1: x^2 + y^2 = a^2$, $c_2: x^2 + y^2 = b^2$, $a < b$



Solution:

$$f(x,y) = \frac{-y}{x^2+y^2}, \qquad g(x,y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial g}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^2+y^2)(-1) + (y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\oint_C [f(x,y)dx + g(x,y)dy] = \iint_R \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA$$

$$\oint_C \left[\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right] = \int_0^{2\pi}\int_a^b \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right] r dr\, d\theta$$

=0

Exercise 1: Verify(حقق) Green's theorem for $f(x,y) = 2xy$, $g(x,y) = -3xy$ on the rectangle $x = 3, x = 5, y = 1, y = 3$

Ans. =-56

Exercise 2: Use Green's theorem to evaluate the line integral

$\oint_C y^2dx + xdy$, on two curves $y = x^2$, $y = \sqrt{x}$