Inverse Laplace Transforms

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2.1. INTRODUCTION

We have already noted that the main purpose of studying Laplace transforms is for solving various types of differential equations. During the process of solving a differential equation, we shall also require to find a function when its Laplace transform is known. This is the reverse process of finding the Laplace transform of a function. In the present chapter, we shall learn to find the function whose Laplace transform is known.

2.2. INVERSE LAPLACE TRANSFORM OF A FUNCTION

Let f be a real valued function of the real variable t, defined for $t \geq 0$. Let the Laplace transform F(s) of $f(t)$ exists. Therefore the infinite integral $\int_0^\infty e^{-st} f(t)\,dt$ exists and equals F(s). The function $f(t)$ is called the **inverse Laplace transform** of the function F(s) and we write $L^{-1}(F(s)) = f(t)$.

In other words, $L^{-1}(F(s))$ is that function whose Laplace transform is the function F(s).

For example,

(i) $L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}$, because $L(e^{at}) = \dfrac{1}{s-a}$

(ii) $L^{-1}\left(\dfrac{s}{s^2+9}\right) = \cos 3t$, because $L(\cos 3t) = \dfrac{s}{s^2+9}$

2.3. EXISTENCE AND UNIQUENESS OF INVERSE LAPLACE TRANSFORM

A given function F(s) of s may or may not have its inverse Laplace transform. So far as the uniqueness of inverse Laplace transforms, we have the following result :

If $f_1(t)$ and $f_2(t)$ be two continuous functions for $t \geq 0$ having the same Laplace transform F(s) i.e. $L^{-1}(F(s)) = f_1(t)$ and $L^{-1}(F(s)) = f_2(t)$, then $f_1(t) = f_2(t) \quad \forall \ t \geq 0$.

We accept this result without proof.

Illustration : The functions $f_1(t) = 1$

and

$$f_2(t) = \begin{cases} 1, & 0 \leq t < 4 \\ 5, & t = 4 \\ 1, & t > 4 \end{cases}$$

have the same Laplace transform $\frac{1}{s}$. Here the above result is not applicable because the function $f_1(t)$ is continuous for $t \geq 0$ but the function $f_2(t)$ is not continuous for $t \geq 0$.

2.4. ELEMENTARY INVERSE LAPLACE TRANSFORM FORMULAE

In this section, we shall find some elementary inverse Laplace transform formulae.

1. We have $$\mathrm{L}(1) = \frac{1}{s}, \; s > 0$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{1}{s}\right) = 1, \; s > 0}.$$

2. We have $$\mathrm{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, \; a > -1, \; s > 0.$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{\Gamma(a+1)}{s^{a+1}}\right) = t^a, a > -1, \; s > 0}.$$

3. We have $$\mathrm{L}(t^n) = \frac{n!}{s^{n+1}}, \; n = 0, 1, 2, \ldots\ldots; s > 0$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n, \; n = 0, 1, 2, \ldots\ldots; s > 0}.$$

4. We have $$\mathrm{L}(e^{at}) = \frac{1}{s-a}, \; s > a.$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{1}{s-a}\right) = e^{at}, \; s > a}.$$

5. We have $$\mathrm{L}(\sinh at) = \frac{a}{s^2 - a^2}, \; s > |a|$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{a}{s^2 - a^2}\right) = \sinh at, \; s > |a|}.$$

6. We have $$\mathrm{L}(\cosh at) = \frac{s}{s^2 - a^2}, \; s > |a|$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{s}{s^2 - a^2}\right) = L(\cosh at), \; s > |a|}.$$

7. We have $$\mathrm{L}(\sin at) = \frac{a}{s^2 + a^2}, \; s > 0$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at, \; s > 0}.$$

8. We have $$\mathrm{L}(\cos at) = \frac{s}{s^2 + a^2}, \; s > 0.$$

$\therefore$ By definition, $$\mathbf{L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at, \; s > 0.}$$

Remarks 1. $L^{-1}\left(\frac{\Gamma(a+1)}{s^{a+1}}\right) = t^a \Rightarrow L^{-1}\left(\frac{1}{s^{a+1}}\right) = \frac{t^a}{\Gamma(a+1)}$; $a > -1,\ s > 0$.

$\therefore$ $$\mathbf{L^{-1}\left(\frac{1}{s^a}\right) = \frac{t^{a-1}}{\Gamma(a)},\ a > 0,\ s > 0.}$$

2. $L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n \Rightarrow L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$, $n = 0, 1, 2, \ldots\ldots; s > 0$

$\therefore$ $$\mathbf{L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!},\ n \in N,\ s > 0.}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L(f(t)) = F(s)$, then $L^{-1}(F(s)) = f(t)$.

Rule II. (*i*) $L^{-1}\left(\frac{1}{s}\right) = 1$ (*ii*) $L^{-1}\left(\frac{\Gamma(a+1)}{s^{a+1}}\right) = t^a$, $a > -1$

(*iii*) $L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$ (*iv*) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(*v*) $L^{-1}\left(\frac{a}{s^2 - a^2}\right) = \sinh at$ (*vi*) $L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh at$

(*vii*) $L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at$ (*viii*) $L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the values of :*

(*i*) $L^{-1}\left(\frac{\Gamma(5/2)}{s^{5/2}}\right)$ (*ii*) $L^{-1}\left(\frac{5040}{s^8}\right)$

(*iii*) $L^{-1}\left(\frac{1}{s+5}\right)$ (*iv*) $L^{-1}\left(\frac{3}{s^2-9}\right)$

(*v*) $L^{-1}\left(\frac{s}{s^2+16}\right)$ (*vi*) $L^{-1}\left(\frac{6}{s^2+36}\right)$.

Sol. (*i*) We have $L^{-1}\left(\frac{\Gamma(a+1)}{s^{a+1}}\right) = t^a$.

$\therefore$ $$L^{-1}\left(\frac{\Gamma(5/2)}{s^{5/2}}\right) = t^{\frac{5}{2}-1} = \mathbf{t^{3/2}}.$$

(*ii*) We have $L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$.

$\therefore$ $$L^{-1}\left(\frac{5040}{s^8}\right) = L^{-1}\left(\frac{7!}{s^{7+1}}\right) = \mathbf{t^7}.$$

(*iii*) We have $$L^{-1}\left(\frac{1}{s-a}\right)=e^{at}.$$

$$\therefore \quad L^{-1}\left(\frac{1}{s+5}\right)=L^{-1}\left(\frac{1}{s-(-5)}\right)=\mathbf{e^{-5t}}.$$

(*iv*) We have $$L^{-1}\left(\frac{a}{s^2-a^2}\right)=\sinh at.$$

$$\therefore \quad L^{-1}\left(\frac{3}{s^2-9}\right)=L^{-1}\left(\frac{3}{s^2-3^2}\right)=\mathbf{\sinh 3t.}$$

(*v*) We have $$L^{-1}\left(\frac{s}{s^2+a^2}\right)=\cos at$$

$$\therefore \quad L^{-1}\left(\frac{s}{s^2+16}\right)=L^{-1}\left(\frac{s}{s^2+4^2}\right)=\mathbf{\cos 4t.}$$

(*vi*) We have $$L^{-1}\left(\frac{a}{s^2+a^2}\right)=\sin at$$

$$\therefore \quad L^{-1}\left(\frac{6}{s^2+36}\right)=L^{-1}\left(\frac{6}{s^2+6^2}\right)=\mathbf{\sin 6t.}$$

2.5. LINEARITY OF THE INVERSE LAPLACE TRANSFORM

Theorem. *If f(t) and g(t) be any functions of t for t ≥ 0 such that L(f(t)) = F(s) and L(g(t)) = G(s) and a and b be any constants, then*

$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F(s)) + bL^{-1}(G(s)).$$

Proof. We have $L(f(t)) = F(s)$ and $L(g(t)) = G(s)$.

$$\therefore \quad aF(s) + bG(s) = aL(f(t)) + bL(g(t))$$
$$= L(af(t) + bg(t))$$

$$\therefore \quad L^{-1}(aF(s) + bG(s)) = af(t) + bg(t)$$

or $$\mathbf{L^{-1}(aF(s) + bG(s)) = aL^{-1}(F(s)) + bL^{-1}(G(s))}.$$

Example 2. *Find the value of* $L^{-1}\left(\frac{1}{s+3}+\frac{2}{s+5}+\frac{6}{s^4}\right)$.

Sol. $$L^{-1}\left(\frac{1}{s+3}+\frac{2}{s+5}+\frac{6}{s^4}\right)$$

$$=L^{-1}\left(\frac{1}{s+3}\right)+2L^{-1}\left(\frac{1}{s+5}\right)+L^{-1}\left(\frac{6}{s^4}\right) \qquad \text{(Using linearity)}$$

$$= L^{-1}\left(\frac{1}{s-(-3)}\right) + 2L^{-1}\left(\frac{1}{s-(-5)}\right) + L^{-1}\left(\frac{3!}{s^{3+1}}\right)$$

$$= \mathbf{e^{-3t} + 2e^{-5t} + t^3.}$$

Example 3. *Find the value of* $L^{-1}\left(\frac{s}{4s^2-16} + \frac{9}{s^2+25} + \frac{4s}{9s^2+4} + \frac{1}{4s-1}\right)$.

Sol. $L^{-1}\left(\frac{s}{4s^2-16} + \frac{9}{s^2+25} + \frac{4s}{9s^2+4} + \frac{1}{4s-1}\right)$

$$= L^{-1}\left(\frac{s}{4s^2-16}\right) + L^{-1}\left(\frac{9}{s^2+25}\right) + L^{-1}\left(\frac{4s}{9s^2+4}\right) + L^{-1}\left(\frac{1}{4s-1}\right)$$

$$= \frac{1}{4}L^{-1}\left(\frac{s}{s^2-2^2}\right) + \frac{9}{5}L^{-1}\left(\frac{5}{s^2+5^2}\right) + \frac{4}{9}L^{-1}\left(\frac{s}{s^2+\left(\frac{2}{3}\right)^2}\right) + \frac{1}{4}L^{-1}\left(\frac{1}{s-\frac{1}{4}}\right)$$

$$= \mathbf{\frac{1}{4}\cosh 2t + \frac{9}{5}\sin 5t + \frac{4}{9}\cos\frac{2}{3}t + \frac{1}{4}e^{\frac{1}{4}t}}\ .$$

Example 4. *Find the inverse Laplace transform of the following functions :*

(*i*) $\frac{4s-8}{9-s^2}$ (*ii*) $\frac{2s-5}{4s^2+25}$

Sol. (*i*) $\frac{4s-8}{9-s^2} = -4\left(\frac{s-2}{s^2-9}\right) = -4\cdot\frac{s}{s^2-9} + \frac{8}{3}\cdot\frac{3}{s^2-9}$

$\therefore$ $L^{-1}\left(\frac{4s-8}{9-s^2}\right) = L^{-1}\left(-4\cdot\frac{s}{s^2-9} + \frac{8}{3}\cdot\frac{3}{s^2-9}\right)$

$$= -4L^{-1}\left(\frac{s}{s^2-3^2}\right) + \frac{8}{3}L^{-1}\left(\frac{3}{s^2-3^2}\right)$$

$$= \mathbf{-4\cosh 3t + \frac{8}{3}\sinh 3t.}$$

(*ii*) $\frac{2s-5}{4s^2+25} = \frac{2}{4}\left(\frac{s-\frac{5}{2}}{s^2+\frac{25}{4}}\right) = \frac{1}{2}\cdot\frac{s}{s^2+\frac{25}{4}} - \frac{5}{4}\cdot\frac{1}{s^2+\frac{25}{4}}$

$\therefore$ $L^{-1}\left(\frac{2s-5}{4s^2+25}\right) = L^{-1}\left(\frac{1}{2}\cdot\frac{s}{s^2+(5/2)^2} - \frac{1}{2}\cdot\frac{5/2}{s^2+(5/2)^2}\right)$

$$= \frac{1}{2}L^{-1}\left(\frac{s}{s^2+(5/2)^2}\right) - \frac{1}{2}L^{-1}\left(\frac{5/2}{s^2+(5/2)^2}\right)$$

$$= \mathbf{\frac{1}{2}\cos\frac{5}{2}t - \frac{1}{2}\sin\frac{5}{2}t.}$$