Extrema values
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Definition: A function f of two variables is said to be have a relative maximum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) s.t. $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk.

Definition: A function f two variable is said to be have a relative minimum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) s.t. $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk.

Theorem: If f has a relative extrema value at a point (x_0, y_0), and if the first order partial derivative of f exists at the point, then

$$f_x(x, y) = 0 \quad \& \quad f_x(x, y) = 0$$

Theorem: A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a critical point of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

Theorem: (The second partial test)
Let f be a function of two variables with continuous second order partial derivatives in some disk centered at a critical point (x_0, y_0) and let

$$\boxed{D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0)\right]^2}$$

- If $D > 0$ & $f_{xx} > 0$, then f has a relative minimum at (x_0, y_0),
- If $D > 0$ & $f_{xx} < 0$, then f has a relative maximum at (x_0, y_0),
- If $D < 0$, then f has a saddle point at (x_0, y_0),
- If $D = 0$, then no conclusion can be drawn(لا يمكن استخلاص أي استنتاج).

Example 1: Locate (حدد) all relative extrema and saddle points of the function $f(x,y) = 3x^2 - 2xy + y^2 - 8y$

Solution: $f_x = 6x - 2y = 0$ ……….. 1

$f_y = -2x + 2y - 8 = 0$ …………… 2

From Eq (1) we have $x = \frac{y}{3}$ …………. 3

$$-2x + 2y - 8 = 0 \quad \div 2 \quad \Rrightarrow \quad -x + y - 4 = 0$$

Substitute Eq(3) in above equation, we obtain

$$-\left(\frac{y}{3}\right) + y - 4 = 0 \quad \Rrightarrow \quad -\frac{y}{3} + \frac{3y}{3} = 4 \quad \Rrightarrow \quad \frac{2y}{3} = 4 \Rrightarrow \quad y = \frac{12}{2}$$

$\therefore \; y = 6$ and $x = 2$ $\Rrightarrow \therefore$ (2,6) is critical point

$f_{xx} = 6 > 0\,,\quad f_{yy} = 2 \quad \& \; f_{xy} = -2$

$$D = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - \left[f_{xy}(x_0,y_0)\right]^2$$

$$\therefore \; D = (6)(2) - (-2)^2 = 12 - 4 = 8 > 0$$

$\therefore$ (2,6) is a relative minimum point.

Example 2: Find extrema values for the function

$$f(x,y) = 3x^3 + y^2 - 9x + 4y$$

Solution: $f_x = 9x^2 - 9 = 0$ ……… 1

$f_y = 2y + 4 = 0$ …………… 2

From Eq (1): $9x^2 - 9 = 0 \;\; \div 9 \;\; \Rrightarrow \; x^2 - 1 = 0 \Rrightarrow \; (x-1)(x+1) = 0$
$\therefore x = \pm 1$

From Eq (2): $2y + 4 = 0 \; \div 2 \quad \Rrightarrow \quad y + 2 = 0 \quad \Rrightarrow \quad y = -2$

$\therefore$ there are two critical points $(-1,-2)$ & $(1,-2)$,
$f_{xx} = 18x\,,\quad f_{yy} = 2 \quad \& \quad f_{xy} = 0$

Point(x_0,y_0)	$f_{xx}(x_0,y_0)$	$f_{yy}(x_0,y_0)$	$f_{xy}(x_0,y_0)$	D	Kind of point
$(-1,-2)$	-18	2	0	-36	not extrema value
$(1,-2)$	18	2	0	36	minimum

Example 3: Locate all relative extrema and saddle points of the function $f(x,y) = 4xy - x^4 - y^4$

Solution: $f_x = 4y - 4x^3 = 0$ ……… 1

$f_y = 4x - 4y^3 = 0$ ……………… 2

From Eq (1) we have

$4y - 4x^3 = 0 \quad \div 4 \quad \Rightarrow \quad y - x^3 = 0$

$\Rightarrow \quad y = x^3$ ….. 3

Substitute Eq(3) in Eq.(2), we obtain

$4x - 4(x^3)^3 = 0 \quad \div 4 \quad \Rightarrow \quad x - x^9 = 0 \Rightarrow x(1 - x^8) = 0$

Either: $x = 0 \Rightarrow y = 0$, $\therefore$ (0,0) is critical point

Or: $1 - x^8 = 0 \Rightarrow x = \pm 1 \Rightarrow y = \pm 1$, $\therefore$ (1,1) & $(-1,-1)$ are critical points

$f_{xx} = -12x^2$, $\quad f_{yy} = -12y^2 \quad \& \quad f_{xy} = 4$

Point(x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	D	Kind of point
(0,0)	0	0	4	-16	saddle
(1,1)	-12	-12	4	128	maximum
$(-1,-1)$	-12	-12	4	128	maximum

Example 4: Find extrema values for the function
$f(x,y) = x^2 + y^2 - xy + 9$

Solution: $f_x = 2x - y = 0$ ……… 1

$f_y = 2y - x = 0$ ……………… 2

From Eq (1) we have : $y = 2x$

Substitute the value of y in Eq (2) , we obtain

$2(2x) - x = 0 \quad \Rightarrow \quad 4x - x = 0 \quad \Rightarrow \quad 3x = 0$

$\therefore \quad x = 0 \quad \Rightarrow \quad y = 0$ and (0,0) is critical point.

$f_{xx} = 2$, $\quad f_{yy} = 2 \quad \& \quad f_{xy} = -1$

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$$

$D = (2)(2) - (-1)^2 = 3 > 0$, $\therefore$ the point (0,0) is a minimum point.

Example 5: Find extrema values for the function
$f(x,y) = x^2 + y^2 - 6xy - 8y + 3$

Solution: $f_x = 2x - 6y = 0$ ……… (1)

$f_y = 2y - 6x - 8 = 0$ …… (2)

From Eq (1): $2x - 6y = 0 \quad \div 2 \quad \Rightarrow \; x - 3y = 0 \Rightarrow \; x = 3y$ (3)

Substitute the value of x in Eq. (2) , we obtain
$2y - 6(3y) - 8 = 0 \quad \Rightarrow \quad -16y = 8 \qquad \div 8 \Rightarrow \quad -2y = 1$
$\therefore \; y = -\frac{1}{2} \quad \Rightarrow x = -\frac{3}{2}$, the critical point is $(\frac{-3}{2}\,,\,\frac{-1}{2})$
$f_{xx} = 2\,, \quad f_{yy} = 2 \quad \& \quad f_{xy} = -6$

$$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \left[f_{xy}(x_0,y_0)\right]^2$$

$D = (2)(2) - (-6)^2 = 4 - 36 = -32 < 0,$

$\therefore$ the point $(\frac{-3}{2}\,,\,\frac{-1}{2})$ is a saddle point, therefor, is not extrema value.

Example 6: Find extrema values for the function
$f(x,y) = 3xy - x^2 - y^2 + x - 4y - 10$

Solution: $f_x = 3y - 2x + 1 = 0$ … (1)

$f_y = 3x - 2y - 4 = 0$ …… (2)

$$\begin{aligned} 3y - 2x &= -1 \qquad \times (3) \\ 3x - 2y &= 4 \qquad \times (2) \end{aligned}$$

$$\begin{aligned} 9y - 6x &= -3 \\ 6x - 4y &= 8 \\ \hline 5y &= 5 \end{aligned}$$

$\therefore \; y = 1$, Substitute the value of y in Eq (1) , we obtain
$3(1) - 2x + 1 = 0 \Rightarrow -2x + 4 = 0 \Rightarrow \; x = 2$
the critical point is $(2\,,1)$
$f_{xx} = -2\,, \quad f_{yy} = -2 \quad \& \quad f_{xy} = 3$

$$D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \left[f_{xy}(x_0,y_0)\right]^2$$

$D = (-2)(-2) - (3)^2 = 4 - 9 = -5 < 0,$

$\therefore$ the point $(2\,,1)$ is a saddle point, therefor, is not extrema value.

Example 7: Find extrema values for the function
$f(x,y) = x^3 + y^2 - 3x + 2y - 6$

Solution: $f_x = 3x^2 - 3 = 0$ ……… 1

$f_y = 2y + 2 = 0$ …… 2

From Eq (1): $3x^2 - 3 = 0 \quad \div 3 \;\Rrightarrow\; x^2 - 1 = 0 \Rrightarrow\; x = \pm 1$

from Eq (2) , we obtain
$2y + 2 = 0 \quad \div 2 \Rrightarrow \quad y + 1 = 0 \qquad \Rrightarrow \quad y = -1$
$\therefore$ $(-1,-1)$& $(1,-1)$ are critical points

$f_{xx} = 6x$, $f_{yy} = 2$ & $f_{xy} = 0$

Point(x_0,y_0)	$f_{xx}(x_0,y_0)$	$f_{yy}(x_0,y_0)$	$f_{xy}(x_0,y_0)$	D	Kind of point
$(-1,-1)$	-6	2	0	-12	saddle
$(1,-1)$	6	2	0	12	minimum

Example 8: Determine the smallest distance between the origin point and the surface (السطح) $z^2 = x^2y + 4$.
Solution:
$d = \sqrt{x^2 + y^2 + z^2}$
$d^2 = x^2 + y^2 + z^2$
$f(x,y) = x^2 + y^2 + x^2y + 4$

Note
المسافة بين نقطة الاصل والسطح
$\boldsymbol{d} = \sqrt{(\boldsymbol{x}-\boldsymbol{x_0})^2 + (\boldsymbol{y}-\boldsymbol{y_0})^2 + (\boldsymbol{z}-\boldsymbol{z_0})^2}$

$f_x = 2x + 2xy = 0$ ……… 1 , $f_y = 2y + x^2 = 0$ …… 2

From Eq (1): $2x + 2xy = 0 \quad \div 2 \;\Rrightarrow\; x + xy = 0 \Rrightarrow\; x(1+y) = 0$
Neither $x = 0$ or $y = -1$
from Eq (2) , we obtain $y = -\frac{x^2}{2}$
if $x = 0$ then $y = 0$ and the critical point is (0,0),
if $y = -1$ then $x = \pm\sqrt{-2y}$, $x = \pm\sqrt{2}$ and the critical point are ($\sqrt{2}$,-1) & (- $\sqrt{2}, -1$)
$f_{xx} = 2 + 2y$, $f_{yy} = 2$ & $f_{xy} = 2x$

Point(x_0,y_0)	$f_{xx}(x_0,y_0)$	$f_{yy}(x_0,y_0)$	$f_{xy}(x_0,y_0)$	D	Kind of point
(0,0)	2	2	0	4	minimum
($\sqrt{2}$,-1)	0	2	$2\sqrt{2}$	-8	saddle
(- $\sqrt{2}, -1$)	0	2	-2$\sqrt{2}$	-8	saddle

$z^2 = x^2y + 4 \quad \Rrightarrow z = 0 + 4$ and $d = \sqrt{0 + 0 + 4} = 2$ is the smallest.

Example 9: Determine the dimensions of a rectangular box, the sum of edges(اضلاعه) is 120cm and requiring the maximum volume of this box.

Solution:

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Note

$v = xyz$
$x + y + z = 120$
$z = 120 - x - y$
$v = xy(120 - x - y)$
$v = 120xy - x^2y - xy^2$

$$v_x = 120y - 2xy - y^2 = 0 \quad \ldots\ldots\ldots \quad (1)$$

$$v_y = 120x - x^2 - 2xy = 0 \ldots\ldots \quad (2)$$

From Eq (1): $y(120 - 2x - y) = 0$
Neither $y = 0$ or $y = 120 - 2x$ ….. (3)

from Eq (2) , we obtain: $x(120 - x - 2y) = 0$
Neither $x = 0$ or $x = 120 - 2y$ …………….. (4)

If $y = 120 - 2(120 - 2y) \Rrightarrow y = 120 - 240 + 4y \Rrightarrow -3y = -120$
$\Rrightarrow y = 40$
$\Rrightarrow x = 40$
From $z = 120 - x - y$ then $z = 40$
$\therefore v = xyz = 40.40.40 = 64000$

Exercise: Find three numbers their sum is 60 and their multiplied the largest value.