The chain Rule

Theorem :- If g is differentiable at X and f is differentiable at $g(X)$, then the compsition $(f*g)$ is differentiable at X, moreover :-

$$(f*g)'(X) = f'(g(x) \cdot g'(x))$$

or $y = f(g(x))$ and $u = g(X)$, then

$y = f(u)$ and :-

$$\frac{dy}{dX} = \frac{dy}{du} \cdot \frac{du}{dX}$$

Ex 1. If $y = \sin^2(3X^5 + X^4 - 2)$ Find $\frac{dy}{dx}$

Solu

let $u = g(X) = 3X^5 + X^4 - 2$ and $f(u) = f(g(x)) = \sin^2 u$

$$\therefore \frac{dy}{du} = 2\sin u \cdot \cos u = \sin 2u$$

$$\frac{du}{dX} = 15X^4 + 4X^3$$

$$\therefore \frac{dy}{dX} = \frac{dy}{du} \cdot \frac{du}{dX}$$

$$= \sin 2u \cdot (15X^4 + 4X^3)$$

$$= [\sin 2(3X^5 + X^4 - 2)](15X^4 + 4X^3)$$

2. Find $\frac{dw}{dt}$; if $w = \tan x$ and $x = 4t^3 + t$

Solu

$$\frac{dw}{dt} = \sec^2 x \quad ; \quad \frac{dx}{dt} = 12t^2 + 1$$

$$\therefore \frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

$$= \sec^2 x \, (12t^2 + 1)$$

3. Find $\frac{dy}{dx}$; if $y = \sqrt{u}$ and $u = e^{2x} + \tan x^2$

Solu

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}} \quad ; \quad \frac{du}{dx} = 2e^{2x} + 2x \sec^2 x^2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot (2e^{2x} + 2x \sec^2 x^2)$$

$$= \frac{1}{2\sqrt{e^{2x} + \tan x^2}} \cdot (2e^{2x} + 2x \sec^2 x^2)$$

4. Find $\frac{dy}{dx}$ for $y = \tan^2 v$; $v = e^{\sin x^2}$

Solu $\frac{dy}{dv} = 2\tan v \sec^2 v$; $\frac{dv}{dx} = e^{\sin x^2} \cdot \cos x^2 \cdot 2x$

$$\therefore \frac{dy}{dx} = 2\tan v \sec^2 v \cdot e^{\sin x^2} \cdot \cos x^2 \cdot 2x$$

$$= 2\tan e^{\sin x^2} \sec^2 (e^{\sin x^2}) \cdot 2x \, e^{\sin x^2} \cdot \cos x^2$$

H.w. Using Chain Rule Find $\frac{dy}{dx}$ for the following:

1. $f(u) = u \cdot \tan^3 u$; $u(x) = \sqrt{x}$

2. $g(u) = \sin u$; $u(x) = \sqrt{1 + \cos x}$

3. $h(u) = \frac{1 + \csc u}{1 - \cot u}$; $u(x) = x^2$

4. $f(x) = \sin^3 u$; $u(x) = e^{x^2+3}$

5. $g(x) = (\ln u)^2$; $u(x) = 3^{x^2+1}$

Ex if $y = t - t^3$ and $x = t - t^2$

Find $\frac{d^2y}{dx^2}$ using chain rule

Solu $y = t - t^3 \Rightarrow \frac{dy}{dt} = 1 - 3t^2$

$x = t - t^2 \Rightarrow \frac{dx}{dt} = 1 - 2t \Rightarrow \frac{dt}{dx} = \frac{1}{1-2t}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= 1 - 3t^2 \cdot \frac{1}{1-2t} \Rightarrow \frac{dy}{dx} = \frac{1-3t^2}{1-2t}$$

$$\frac{d^2y}{dx^2} = \quad 0$$

Higher Derivatives

If the derivatives f' of a function f is itself differentable, then the derivate of f' is denoted by f'' and is called the second derivative of f. As long as we have differentiability, we can contine the process of differentiating derivatives to obtain third, fourth, fifth and even higher derivative of f.

The successive derivatives of f are denoted by:-

$$f', f'', f''', \ldots, f^{(n)} \quad \text{or} \quad \frac{dy}{dx}; \frac{d^2y}{dx^2}, \ldots$$

Ex Find $f^{(5)}$ for the function:-

1. $f = 3x^4 - 2x^3 + x^2 - 4x + 2$

Solu

$$f' = 12x^3 - 6x^2 + 2x - 4$$

$$f'' = 36x^2 - 12x + 2$$

$$f''' = 72x - 12$$

$$f'''' = 72$$

$$f^{(5)} = 0$$

2. $f(x) = \frac{1}{x}$ Find $f^{(3)}(x)$

$$f'(x) = \frac{-1}{x^2} \; ; \; f''(x) = \frac{2}{x^3} \; , \; f'''(x) = \frac{-6}{x^4}$$

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3. Find $\left.\frac{d^2y}{dx^2}\right|_{x=1}$ where $y = 6x^5 - 4x^2$

Solu

$$\frac{dy}{dx} = 30x^4 - 8x$$

$$\frac{d^2y}{dx^2} = 120x^3 - 8 \Big|_{x=1}$$

$$\frac{d^2y}{dx^2} = 112$$

4. $f(x) = \frac{x^2-1}{x^4+1}$

$$f'(x) = \frac{(x^4+1)(2x) - (x^2+1)(4x^3)}{(x^4+1)^2}$$

$$= \frac{2x^5 + 2x - 4x^5 + 4x^3}{(x^4+1)^2}$$

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2}$$

5. $g(x) = \frac{\sin x}{1+\cos x}$

$$g'(x) = \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

6. $h(x) = \sec x \cdot \tan x$

$h'(x) = \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$

$= \sec^3 x + \tan^2 x \sec x$

$= \sec^3 x + \sec x (\sec^2 x - 1)$

$= \sec^3 x + \sec^3 x - \sec x$

$= 2\sec^3 x - \sec x$

or

$= \sec x (\tan^2 - 1) + \sec x \tan^2 x$

$= 2 \sec x \cdot \tan^2 x - \sec x$

H.W

Find the derivative for the following :-

1. $f(x) = \sqrt{x} + \frac{1}{x}$
2. $f(x) = \ln \frac{x^2+1}{x^2+2}$
3. $g(x) = e^{x^2} \cdot \sin x$
4. $g(x) = \sec(e^{x^2})$
5. $g(x) = (1 + \cos^2 x)^{1/2}$
6. $h(x) = \tan x^2 / \sqrt{x+1}$
7. $h(x) = x \cos^2 x$
8. $h(x) = (x^2 + 2x)^8$

4. Show that $y = x^3 + 3x + 1$ satisfies

$$y''' + x y'' - 2y' = 0$$

Solu

$$y' = 3x^2 + 3$$
$$y'' = 6x$$
$$y''' = 6$$

$$\because\ y''' + x y'' - 2y' = 0$$
$$6 + x\,6x - 2(3x^2 + 3) = 0$$
$$6 + 6x^2 - 6x^2 - 6 = 0$$

5. Find $y''(\frac{\pi}{4})$ if $y = \sec x$

Solu

$$y'(x) = \sec x \cdot \tan x$$
$$y''(x) = \sec x \cdot \sec^2 x + \tan x \cdot \tan x \sec x$$
$$= \sec^3 x + \sec x \tan^2 x$$

$$y''(\tfrac{\pi}{4}) = \sec^3(\tfrac{\pi}{4}) + \sec(\tfrac{\pi}{4}) \tan^2(\tfrac{\pi}{4})$$

$$= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2}$$

6. Find y'' for

i) $y = x \sin x - 3\cos x$

$$y' = [x \cdot \cos x + \sin x\,(1)] - 3(-\sin x)$$
$$= x\cos x + \sin x + 3\sin x$$
$$= x\cos x + 4\sin x$$

$$y'' = [x(-\sin x) + \cos x\,(1)] + 4\cos x$$
$$= -x\sin x + \cos x + 4\cos x$$
$$= 5\cos x - x\sin x$$

ii) $y = e^{\tan x^2}$

$y' =$

$y' = e^{\tan x^2} \cdot \sec^2 x^2 \cdot 2x$

$y'' = e^{\tan x^2} \cdot [\sec^2 x^2 \cdot 2 + 2x \cdot 2\sec x^2 \cdot \sec x^2 \cdot \tan x^2]$

$+ \sec^2 x^2 \cdot 2x \cdot e^{\tan x^2} \cdot \sec^2 x^2 \cdot 2x]$

$= e^{\tan x^2} \cdot \sec^2 x^2 (2 + 4\sec x^2 \cdot \tan x^2 + 4x \sec^2 x\, x^3)$

H.W.

Find y''' for the following:

1. $y = \csc x$
2. $y = (x^2 + 1) \cdot \sec x$
3. $y = \dfrac{\sin x}{x}$
4. $y = \ln x^2$
5. $y = e^{\cos x}$
6. $y = a^{\cot x}$