6th Lecture

Exterior Set المجموعة الخارجية

Definition (1.7): Let (X, τ) be a topological space and $E \subset X$, we define the **exterior** of a set E, denoted by E^e or $e(E)$ as follow:

$$E^e = \{x \in X : y \in E^{c^\circ}\} \text{ , i.e. } E^e = E^{c^\circ}$$

Example (1.11):

Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{a, c\}, \{a, c, d\}, \{b, c, d, e\}, X\}$.

Find the exterior of the sets $A = \{a, b, d\}$, $B = \{c, d, e\}$, and $E = \{b, e\}$.

Solution:

$A = \{a, b, d\} \Rightarrow A^c = \{c, e\}$

The open sets contained in A^c is $\emptyset$

$\Rightarrow A^{c^\circ} = \bigcup_{\forall G \subset A^c} G = \bigcup \emptyset = \emptyset$

$\Rightarrow A^e = \emptyset$

We have

$B = \{c, d, e\} \Rightarrow B^c = \{a, b\}$

The open sets contained in B^c are $\emptyset$ and $\{a\}$

$\Rightarrow B^{c^\circ} = \bigcup_{\forall G \subset B^c} G = \emptyset \cup \{a\} = \{a\}$

$\Rightarrow B^e = \{a\}$

We have

$E = \{b, e\} \Rightarrow E^c = \{a, c, d\}$

The open sets contained in E^c are $\emptyset$, $\{a\}$, $\{a, c\}$ and $\{a, c, d\}$

$\Rightarrow E^{c^\circ} = \bigcup_{\forall G \subset E^c} G = \emptyset \cup \{a\} \cup \{a, c\} \cup \{a, c, d\} = \{a, c, d\}$

$\Rightarrow\ E^e = \{a, c, d\}$

Corollary (1.3): If A, B are subsets of (X, τ). Then

(i) $\emptyset^e = X$

(ii) $A^e \subset A^c$

(iii) $B^e = B^{e^{c^e}}$

(iv) $(A \cup B)^e = A^e \cap B^e$

Proof:

(i) $\emptyset^e = \emptyset^{c^\circ} = X^\circ = X$

(ii) $A^e = A^{c^\circ} \subset A^c \Rightarrow A^e \subset A^c$

(iii) $B^{e^{c^e}} = \left(\left(B^{c^\circ}\right)^c\right)^{c^\circ} = \left(B^{c^\circ}\right)^\circ = B^{c^\circ} = B^e$

$\therefore\ B^e = B^{e^{c^e}}$

(iv) $(A \cup B)^e = (A \cup B)^{c^\circ}$

$= (A^c \cap B^c)^\circ$

$= A^{c^\circ} \cap B^{c^\circ} = A^e \cap B^e$

$\therefore\ (A \cup B)^e = A^e \cap B^e$

Boundary of a Set حدود المجموعة

Definition (1.8): Let (X, τ) be a topological space and $E \subset X$, we define the **boundary** of a set E, denoted by E^b or $b(E)$ as follow:

$$E^b = (E^\circ \cup E^e)^c$$

Note (1.1): $E^b = (E^\circ \cup E^e)^c = E^{\circ^c} \cap E^{e^c} = E^{\circ^c} \cap E^{c^{\circ^c}} = E^{\circ^c} \cap \bar{E} = \bar{E} - E^\circ$

$\Rightarrow\ E^b = \bar{E} - E^\circ$

Example (1.12):

Let $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a\}, \{a, c\}, \{a, c, d\}, \{b, c, d, e\}, X\}$.

Find the boundary of the sets $A = \{a, b, c\}$, $B = \{b, d, e\}$.

Solution:

$A = \{a, b, c\}$

The closed sets are X, $\{b, c, d, e\}$, $\{b, d, e\}$, $\{b, e\}$, $\{a\}$, $\emptyset$

$\bar{A} = \bigcap_{\forall\, F \supset A} G,$ where F is closed

$= X$

$A^\circ = \bigcup_{\forall\, G \subset A} G$, where G is open

$= \emptyset \cup \{a\} \cup \{a, c\} = \{a, c\}$

Hence

$A^b = \bar{A} - A^\circ$

$= X - \{a, c\} = \{b, d, e\}$

We have

$B = \{b, d, e\}$

$\bar{B} = \bigcap_{\forall\, F \supset B} G,$ where F is closed

$= X \cap \{b, c, d, e\} \cap \{b, d, e\} = \{b, d, e\}$

$B^\circ = \bigcup_{\forall\, G \subset B} G$, where G is open

$= \cup \emptyset = \emptyset$

Hence

$B^b = \bar{B} - B^\circ$

$= \{b, d, e\} - \emptyset = \{b, d, e\}$
