Integration of trigonometric Functions:-

a) Integration of odd Functions

We have: $\sin^2 x + \cos^2 x = 1$, Then:

$$\sin^2 x = 1 - \cos^2 x$$
$$\cos^2 x = 1 - \sin^2 x$$

Therefore, the integral of Functions contaning odd exponents of sin and cos is as follows:-

$$\int \sin^m x \cos^n x \, dx$$

where : m is odd and n is even

Then the integration of these functions is as follows:-

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x \cos x \cos^{n-1} x \, dx$$

Ex

1. $\int \sin^3 x \cos^2 x \, dx$

Sol:

$$= \int \sin x \sin^2 x \cos^2 x \, dx$$

$$= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$$

$$= \int \sin x \cos^2 x - \sin x \cos^4 x \, dx = \frac{-\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

2. $\int \cos^5 x \sin^2 x \, dx$

Solu

$$= \int \cos x \cos^4 x \sin^2 x \, dx$$

$$= \int \cos x (\cos^2 x)^2 \sin^2 x \, dx$$

$$= \int \cos x (1 - \sin^2 x)^2 \sin^2 x \, dx$$

$$= \int \cos x (1 - 2\sin^2 x + \sin^4 x) \sin^2 x \, dx$$

$$= \int (\cos x \sin^2 x - 2 \cos x \sin^4 x + \cos x \sin^5 x) \, dx$$

$$= \frac{\sin^3 x}{3} - 2 \frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

b) Integration of even functions

The following relationships are used:

① $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

② $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

Ex 1. $\int \sin^2 x \, dx$

$$= \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\left[\int dx - \int \cos 2x \, dx\right]$$

$$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

2. $\int \sin x \cos^3 x \, dx$

$$= \int \sin x \cos x (1-\sin^2 x)\,dx$$

$$= \int (\sin x \cos x - \cos x \sin^3 x)\,dx$$

$$= \frac{\sin^2 x}{2} + \frac{\sin^4 x}{4} + c$$

H.w. $\int \sin^4 x \, dx$; $\int \cos^2 x \, dx$; $\int \sin^6 x \, dx$, $\int \cos^6 x \, dx$

3. $\int \sin^4 x \, dx$

Solu $= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$= \int \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)\, dx$

$= \frac{1}{4}\left(\int dx - 2\int \cos 2x \, dx + \int \cos^2 2x \, dx\right)$

$= \frac{1}{4}\left(x - 2\,\frac{1}{2}\sin 2x + \int \frac{1}{2}(1+\cos 4x)\, dx\right.$

$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x - \frac{1}{32}\sin^2 4x + c$

4. $\int \cos^4 x \, dx$

$= \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx = \int \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)\, dx$

$= \int \frac{1}{4}\, dx + \int \frac{2}{4}\cos 2x \, dx + \int \frac{1}{4}\cos^2 2x \, dx$

$= \frac{1}{4}x + \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x + c$

Some laws of trigonometric ratios:

1. $\sin x \cos x = \frac{1}{2}\sin 2x$

2. $\sin x \cos y = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y)$

3. $\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$

4. $\cos x \cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$

Ex

1. $\int \sin 4x \sin x \, dx$

Solu

$$= \int \frac{1}{2}\left[\cos(4x - x) - \frac{1}{2}\cos(4x + x)\right] dx$$

$$= \int \left(\frac{1}{2}\cos 3x - \frac{1}{2}\cos 5x\right) dx$$

$$= \frac{-1}{6}\sin x - \frac{1}{10}\sin 5x + c$$

2. $\int \sin 3x \cos 4x \, dx$

Solu

$$= \int \frac{1}{2}\sin(3x + 4x) + \frac{1}{2}\sin(3x - 4x) \, dx$$

$$= \int \frac{1}{2}\sin 7x + \frac{1}{2}\sin(-x) \, dx$$

$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos(-x) + c$$

Integration of the tan and cot functions:-

It is done using the two identities:-

① $\tan^2 x = \sec^2 x - 1$

$$\sin^2 x + \cos^2 x = 1 \qquad \div \cos^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

② $\cot^2 x = \csc^2 x - 1$

$$\sin^2 x + \cos^2 x = 1 \qquad \div \sin^2 x$$

$$1 + \cot^2 x = \csc^2 x$$