(4) Since the $\dim(X) = n$, then the set $\{x_i : 1 \le i \le n+1\}$ is linearly dependent, that is mean

$$\alpha x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n + \alpha_{n+1} x_{n+1} = 0$$

there exist i, $1 \le i \le n+1$ such that $\alpha_i \neq 0$

now

$$T(\alpha_1 x_1 + \cdots \alpha_n x_n + \alpha_{n+1} x_{n+1}) = T(0) = 0$$

$$T(\alpha_1 x_1 + \cdots \alpha_n x_n + \alpha_{n+1} x_{n+1}) = \alpha_1 T(x_1) + \cdots \alpha_{n+1} T(x_{n+1})$$

$$= \alpha_1 y_1 + \cdots \alpha_n y_n + \alpha_{n+1} y_{n+1}$$

$$= 0$$

The linear combination of $n+1$ element are equal zero but not all α_i are equal to zero, then this mean linear combination are linearly dependent, we conclude that $R(T)$ has no linearly independent subset of $n+1$ or more elements. That is mean

$$\dim(R(T)) \le n.$$

(3) Let $x_1, x_2 \in N(T)$, Then $T(x_1) = 0$, $T(x_2) = 0$

Let $\alpha, \beta \in K$, we want to show that $\alpha_1 x_1 + \alpha_2 x_2 \in N(T)$

$$T(\alpha_1 x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$$

$$= \alpha \cdot 0 + \beta \cdot 0$$

$$= 0$$

This is show that $\alpha x_1 + \beta x_2 \in N(T)$

is a subspace of X.

Note if T is linear operator then

$$T(0) = 0$$

$$T(0+0) = T(0) + T(0)$$

$$T(0) = T(0) + T(0)$$

$$T(0) - T(0) = T(0)$$

$$0 = T(0)$$

Inverse operator.

Before we start with definition of inverse operator. we first remember that an operator

$T : D(T) \longrightarrow Y$ تباين

is said to be injective or one-to-one

if for any different points in the domain have different images. that is if for any $x_1, x_2 \in D(T)$,

$$x_1 \neq x_2 \longrightarrow f(x_1) \neq f(x_2)$$

equivalently

$$f(x_1) = f(x_2) \longrightarrow x_1 = x_2$$ كامل

The operator T is said to surjective, if for all $y \in Y$, $\exists\, x \in D(T)$, $T(x) = y$ or $R(T) = Y$

Thee operator T is said to bijective if T is one-to-one and onto.

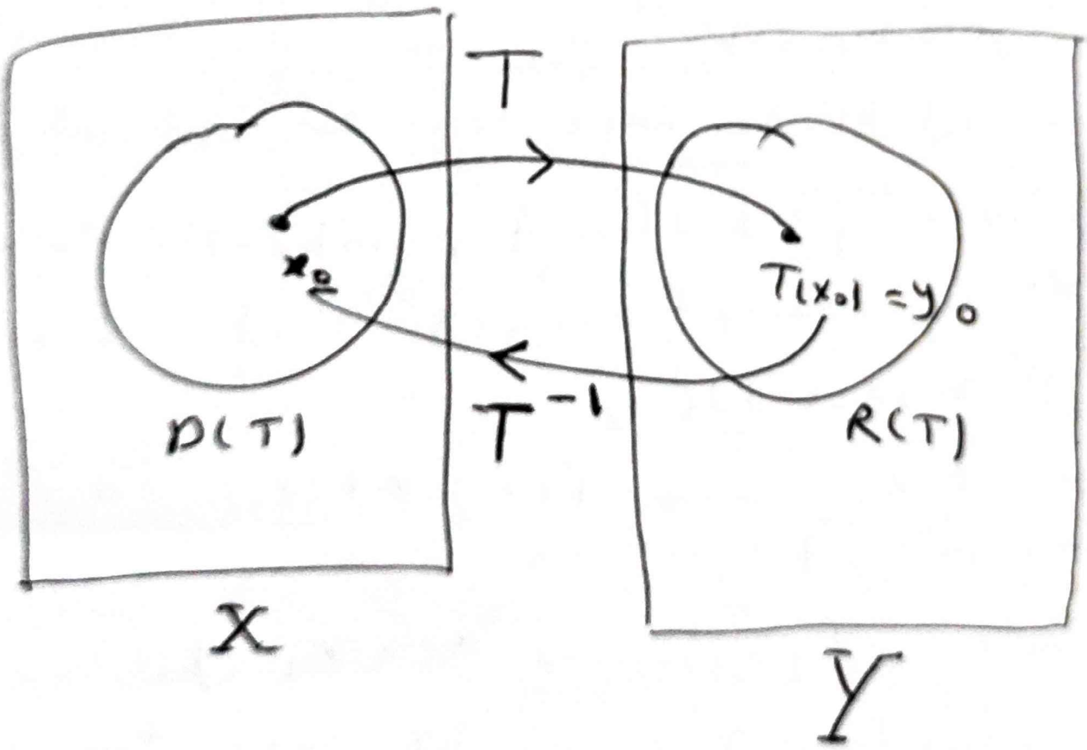
(18) Inverse operator

Let T be an operator for $D(T)$ into Y
$T : D(T) \longrightarrow Y$, the operator defined from the $R(T)$ into $D(T)$ is called inverse operator of T and denoted it by T^{-1}.

$$T^{-1} : R(T) \longrightarrow D(T).$$

for $y \in R(T)$, then exist $x \in D(T)$ such that $T(x) = y$ we define T^{-1} as

$$T^{-1}(y) = x$$



Note :-

1) The operator T^{-1} is exist if and only if T is one-to-one.

2) $T\,T^{-1}(y) = y = I_{R(T)}(y)$

$I_{R(T)} : R(T) \longrightarrow R(T).$

3) $T^{-1}\,T(x) = x = I_{D(T)}(x)$

$I_{D(T)} : D(T) \longrightarrow D(T).$

we see the operator $T\,T^{-1}$ is different from the operator $T^{-1}T$ operator.

Example

1) let $I : X \longrightarrow X$ be the identity operator Then $I^{-1} = I$ itself the inverse operator.

2) Let $T : R^2 \longrightarrow R^2$ such that

$T(x) = T(x_1, x_2) = (x_2, x_1) = y$

Let $v, w \in R^2$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ such that.

$T(v) = T(w)$

$(v_2, v_1) = (w_2, w_1) \longrightarrow v_2 = w_2\,,\ v_1 = w_1$

$v = (v_1, v_2)\,,\ w = (w_1, w_2).$

so T is one-to-one then T^{-1} is exist.

$T^{-1}(y) = T^{-1}(x_2, x_1) = (x_1, x_2) = x$.

(18) Theorem (inverse operator)

Let X, Y be a vector spaces over the same field. Let $T: D(T) \longrightarrow Y$ be a linear operator with domain $D(T) \subseteq X$ and range $R(T) \subseteq Y$. Then

① The inverse $T^{-1}: R(T) \longrightarrow D(T)$ exists if and only if $T(x) = 0 \rightarrow x = 0$.

② If T^{-1} is exists, it is a linear operator.

③ If $\dim D(T) = n < \infty$ and T^{-1} is exists Then $\dim R(T) = \dim(D(T))$

proof: ① suppose $T(x) = 0 \longrightarrow x = 0$
we to show that T^{-1} is exists, we must show that T is one-to-one, so let $x_1, x_2 \in X$ such that

$$T(x_1) = T(x_2)$$
$$T(x_1) - T(x_2) = 0$$
$$T(x_1 - x_2) = 0 \longrightarrow x_1 - x_2 = 0$$
$$x_1 = x_2$$

So T is one-to-one, therefore T^{-1} is exists.

Converse let T^{-1} is exists
that is mean T is one-to-one
let $T(x)=0$, since $T(0)=0$
so $T(x)=T(0)$ $\because$ T is one-to-one
$x=0$

(2) we assume that T^{-1} is exists and show that T^{-1} is linear, The domain of T^{-1} is $R(T)$ and $R(T)$ is a vector space

let $y_1, y_2 \in R(T)$, then exists $x_1, x_2 \in D(T)$
such that $y_1 = T(x_1)$, $y_2 = T(x_2)$ so $T^{-1}(y_1) = x_1$, $T^{-1}(y_2) = x_2$

Let $\alpha, \beta \in K$

$$\alpha y_1 + \beta y_2 = \alpha T(x_1) + \beta T(x_2) \quad \because T \text{ is linear}$$
$$= T(\alpha x_1 + \beta x_2)$$

$$\text{So } T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2$$
$$= \alpha T^{-1}(y_1) + \beta T^{-1}(y_2)$$

and this prove that T^{-1} is linear.

3 It is clearly that from theorem

"Let T be a linear operator Then : if $\dim X = n$ then $\dim R(T) \leq n$ "

So $\dim R(T) \leq \dim D(T)$ —— (1)

T^{-1} is also an operator by the same theorem

$\dim R(T^{-1}) \leq \dim D(T^{-1})$

but $R(T^{-1}) = D(T)$

and $D(T^{-1}) = R(T)$

so $\dim(D(T)) \leq \dim R(T)$ —— (2)

from (1) & (2) we get that

$\dim(R(T)) = \dim D(T)$ ..

Space of linear operators

Let X, Y be vector spaces over the same field K, all linear operator from vectors space X into vector space Y is denoted by $L(X,Y)$.

$L(X,Y) = \{ T : T : X \longrightarrow Y \text{ is linear} \}$

we have $(L(X,Y), +, \cdot, K)$ is a vector space with addition operation defined as

$+ : L(X,Y) \times L(X,Y) \longrightarrow L(X,Y)$

for $T_1, T_2 \in L(X,Y)$

$(T_1 + T_2)(x) = T_1(x) + T_2(x)$ for $x \in X$

The saclar multiplication on $L(X,Y)$ over K is defined as

for $T \in L(X,Y)$, $\alpha \in K$

$(\alpha \cdot T)(x) = \alpha \cdot T(x)$

(22) At first we to show that $(L(X,Y),+)$ is commutative group.

(A_1) we to show that $L(X,Y)$ is closed under addition or we say that $L(X,Y)$ with addition represent a binary operation

Let $T_1, T_2 \in L(X,Y)$ we to show $T_1+T_2 \in L(X,Y)$ that is mean must prove that T_1+T_2 is linear operator

Let $x_1, x_2 \in X$, $\alpha, \beta \in K$

$$(T_1+T_2)(\alpha x_1+\beta x_2) = T_1(\alpha x_1+\beta x_2) + T_2(\alpha x_1+\beta x_2)$$

$\because T_1, T_2$ are linear operators

$$= \alpha T(x_1) + \beta T_1(x_2) + \alpha T_2(x_1) + \beta T_2(x_2)$$

$$= \alpha(T(x_1) + T_2(x_1)) + \beta(T_1(x_2) + T_2(x_2))$$

$$= \alpha(T_1+T_2)(x_1) + \beta(T_1+T_2)(x_2)$$

(A_2) associative law

Let T_1, T_2, T_3

$$[(T_1+T_2)+T_3](x) = (T_1+T_2)(x) + T_3(x)$$

$$= [T_1(x) + T_2(x)] + T_3(x)$$

$\because Y$ ~~is as~~ has associative addition th.

$$= T_1(x) + [T_2(x) + T_3(x)]$$

$$= T_1(x) + (T_2+T_3)(x)$$

$$= [T_1+(T_2+T_3)](x) \quad \forall x \in X$$

$$\therefore (T_1+T_2)+T_3 = T_1+(T_2+T_3)$$

A_3 Identity element is the zero operator for addition

$$(T+O)(x) = T(x) + O(x)$$
$$= T(x) + 0$$
$$= T(x) \quad \forall x \in X$$

we get that $T + O = T$

similarly $(O+T)(x) = T(x)$ we get $O + T = T$

$\therefore O$ is the identity element. $\forall T \in L(X,Y)$

A_4 for any $T \in L(X,Y)$, there exists $-T \in L(X,Y)$

The inverse element for addition

$$(T + -T)(x) = T(x) + (-T)(x)$$
$$= T(x) + -T(x)$$
$$= 0 \quad \forall x \in X$$
$$= O(x)$$

So $T + -T = O$

similarly we get that $-T + T = O \quad \forall T \in L(X,Y)$

A_5 Let $T_1, T_2 \in L(X,Y)$ $\quad T_1(x), T_2(x) \in Y$

$$(T_1 + T_2)(x) = T_1(x) + T_2(x)$$
$$= T_2(x) + T_1(x) \quad \text{since } (Y,+) \text{ is commutative}$$
$$= (T_2 + T_1)(x) \quad \forall x \in X$$

$\therefore T_1 + T_2 = T_2 + T_1$

$\forall T_1, T_2 \in L(X,Y)$

So $(L(X,Y), +)$ is commutative group

(24)

(5) we want to show that $L(X,Y)$ is closed under scalar multiplication, so must prove that for any $\alpha \in K$, $T \in L(X,Y)$ implies $\alpha T \in L(X,Y)$

To prove $\alpha T \in L(X,Y)$ must prove αT is linear operator for $x_1, x_2 \in X$

$$(\alpha T)(x_1+x_2) = \alpha \cdot T(x_1+x_2) \quad T \text{ is linear}$$
$$= \alpha \cdot (T(x_1) + T(x_2))$$
$$= \alpha \cdot T(x_1) + \alpha \cdot T(x_2)$$
$$= (\alpha \cdot T)(x_1) + (\alpha T)(x_2)$$

Let $\beta \in K$, $x \in X$

$$(\alpha T)(\beta x) = \alpha T(\beta X) \quad \because T \text{ is linear operator}$$
$$= \alpha \cdot \beta T(X)$$
$$= \beta \cdot \alpha T(X)$$
$$= \beta \cdot (\alpha T)(x)$$

$\therefore$ αT is linear operator.

S_2 $[\alpha \cdot (T_1 + T_2)](x) = \alpha \cdot (T_1 + T_2)(x)$

$= \alpha(T_1(x) + T_2(x))$

$= \alpha \cdot T_1(x) + \alpha \cdot T_2(x)$

$= (\alpha T_1)(x) + (\alpha T_2)(x)$

$= (\alpha T_1 + \alpha T_2)(x) \quad \forall x \in X$

so $\alpha \cdot (T_1 + T_2) = \alpha T_1 + \alpha T_2$

S_3 $[(\alpha + \beta) T](x) = (\alpha + \beta) \cdot T(x)$

$= \alpha \cdot T(x) + \beta \cdot T(x)$

$= (\alpha \cdot T + \beta \cdot T)(x) \quad \forall x \in X$

$(\alpha + \beta) T = \alpha \cdot T + \beta \cdot T$

S_4 $[(\alpha \cdot \beta) T](x) = (\alpha \cdot \beta) \cdot T(x)$

$= \alpha \cdot [\beta T(x)]$

$= \alpha \cdot (\beta T)(x) \quad \forall x \in X$

$(\alpha \cdot \beta) T = \alpha \cdot (\beta T)$

S_5 $(1 \cdot T)(x) = 1 \cdot T(x)$
$= T(x) \quad \forall x \in X$

$1 \cdot T = T$

so $L(X, Y)$ is a vector space over K

when $X = Y$ we always write it

$L(X, Y) = L(X)$.

(26)

Composite (product)

~~Let~~ ~~...~~

Let $T: X \longrightarrow Y$ and $S: Y \longrightarrow Z$ be two linear operators where X, Y, Z are vector spaces. we define the ~~composit~~ operator $ST: X \longrightarrow Z$ to be composite between S, T as we shown below

$$(ST)(x) = S(T(x))$$

Lemma Let $T: X \longrightarrow Y$, $S: Y \longrightarrow Z$ be two linear operators when X, Y, Z are vectorspaces Then.

① ST are linear.

② If S, T is bijective. Then $(ST)^{-1}$ is exists and it is linear

$$(ST)^{-1} = T^{-1} S^{-1}$$

proof: ① Let $x_1, x_2 \in X$,

$$(ST)(x_1 + x_2) = S(T(x_1 + x_2))$$
$$= S(T(x_1) + T(x_2))$$
$$= ST(x_1) + ST(x_2)$$

for $\alpha \in X$ ~~...~~

$$(ST)(\alpha x) = ST(\alpha x) = S(\alpha T(x)) = \alpha . ST(x)$$
$$= \alpha . (ST)(x) \quad \forall x \in X$$

ST is linear operator.

$\because S, T$ is a bijection operator

so ST is also bijection then we have

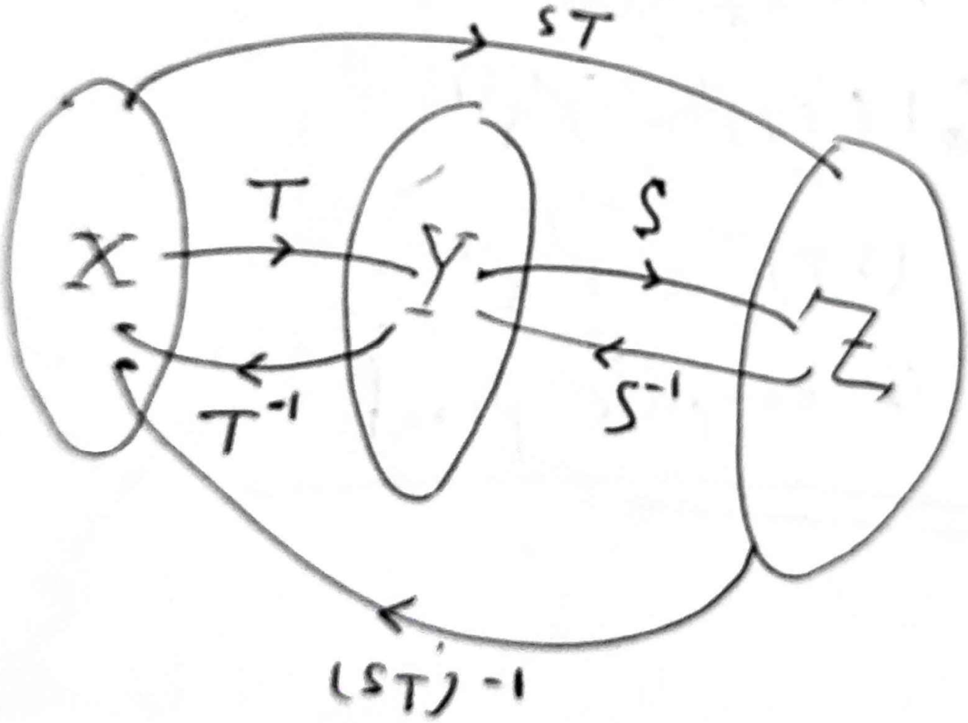
$(ST)^{-1}$ is exist and linear

$\because S, T$ is bijection $\therefore$ so S^{-1}, T^{-1} is exist and linear.

by using Theorem

* Let X, Y be a vector spaces over the same field. Let $T: D(T) \longrightarrow Y$ be a linear operator with domain $D(T) \subseteq X$ and $R(T) \subseteq Y$. Then If T^{-1} is exists, it is a linear operator //

now $ST(ST)^{-1} = I_Z$



$(ST)(ST)^{-1} = I_Z$

(28)

$$(ST)(ST)^{-1} = I_Z$$

by taking S^{-1} for two side

$$(S^{-1}S)\,T(ST)^{-1} = S^{-1}I_Z$$

since $S^{-1}S = I_Y$

$$(I_Y)\,T(ST)^{-1} = S^{-1}$$

$$T(ST)^{-1} = S^{-1}$$

by taking T^{-1} for two side

$$(T^{-1}T)(ST)^{-1} = T^{-1}S^{-1}$$

since $T^{-1}T = I_X$

$$(I_X)(ST)^{-1} = T^{-1}S^{-1}$$

$$(ST)^{-1} = T^{-1}S^{-1}$$

This is complete proof.

Example: Let $T_1 : R^2 \to R^2$ such that

$T_1(x_1, x_2) = (x_2, x_1)$

$T_2 : R^2 \to R^2$ such that

$T_2(x_1, x_2) = (x_1, 0)$

$\cancel{T_2(x_1, x_2)}$

$$
\begin{aligned}
(T_1 T_2)(x) &= T_1(T_2(x)) \\
&= T_1(T_2(x_1, x_2)) \\
&= T_1((x_1, 0)) \\
&= (0, x_1)
\end{aligned}
$$

$$
\begin{aligned}
(T_2 T_1)(x) &= T_2(T_1(x)) \\
&= T_2(T_1(x_1, x_2)) \\
&= T_2(x_2, x_1) \\
&= (x_2, 0)
\end{aligned}
$$

So $T_1 T_2(x) \neq T_2 T_1(x)$

Therefore $T_1 T_2 \neq T_2 T_1$