2.1 Representation of functionals on Hilbert spaces

one of importance theorem in functional analysis is Riesz's theorem, which say that every bounded linear functional on a Hilbert space can be represent as inner product. for ~~general~~ general Banach space the bounded linear functional are more complicated to be represent.

Theorem (Riesz's Theorem)

Every bounded linear functional f on a Hilbert space X can be represented in terms of the inner product, namly,

$$f(x) = \langle x, z \rangle \quad \forall x \in X$$

when z depends on f is uniquely determined by f and has norm.

$$\|f\| = \|z\|$$

furthermore

$z \perp N(f)$ when $N(f)$ is the null space of f

Note

Let $f: X \longrightarrow K$ be an functional then

$N(f) = \{ x \in X / f(x) = 0 \}$

$N(f)$ is called the null space.

Example Let $X = \mathbb{C}^2$ over $\mathbb{C}$ and $f: X \longrightarrow \mathbb{C}$ be an functional is f satisfies Riez's theorem if is true find z? $f(x) = f(x_1, x_2) = x_1$
$x \in \mathbb{C}^2$, $x = (x_1, x_2)$

Solution

At first we check that f is bounded linear

Let $\alpha, \beta \in \mathbb{C}$, $x, y \in \mathbb{C}^2$ such that
$x = (x_1, x_2)$
$y = (y_1, y_2)$

$$
\begin{aligned}
f(\alpha x + \beta y) &= f(\alpha(x_1, x_2) + \beta(y_1, y_2)) \\
&= f((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)) \\
&= f(\underline{\alpha x_1 + \beta y_1}, \alpha x_2 + \beta y_2)) \\
&= \alpha x_1 + \beta y_1 \\
&= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \\
&= \alpha f(x) + \beta f(y)
\end{aligned}
$$

so f is linear operator.

Let $x \in X$, $x = (x_1, x_2)$. $x_1, x_2 \in \mathbb{C}$

$$|f(x)|^2 = |f(x_1, x_2)|^2 = |x_1|^2$$
$$= x_1 \overline{x_1}$$
$$\leq x_1 \overline{x_1} + x_2 \overline{x_2}$$
$$= \langle (x_1, x_2) / (x_1, x_2) \rangle$$
$$= \langle x / x \rangle$$
$$= \|x\|^2$$

So $|f(x)| \leq \|x\| \quad \forall x \in \mathbb{C}^2$ —— (1)

$\therefore f$ is bounded

So by using Riesz's theorem $\exists\, z \in \mathbb{C}^2$ such that $f(x) = \langle x / z \rangle$

$f(x) = f(x_1, x_2) = x_1$ ~~<x/z>~~

$\langle x / z \rangle = \langle (x_1, x_2) / (z_1, z_2) \rangle$
$= x_1 \overline{z_1} + x_2 \overline{z_2}$

$\therefore f(x) = \langle x / z \rangle$

$x_1 = x_1 \overline{z_1} + x_2 \overline{z_2}$

$x_1 = x_1 (1) + x_2 (0)$

we see when $\overline{z_2} = 0$ and $\overline{z_1} = 1$

$\therefore z = (1, 0)$

$\|z\|^2 = \langle z / z \rangle = \langle (1,0) / (1,0) \rangle$
$= 1^2 + 0 = 1$

~~also $\|f\| = \sup$~~

$\because f$ is bounded we have

$$|f(x)| \leq \|f\| \, \|x\| \quad \forall x \in X$$

so if $x = (1, 0)$ then

$$|f(1, 0)| \leq \|f\| \, \|(1, 0)\|$$

$$\frac{|1|}{\|(1,0)\|} \leq \|f\|$$

$$\frac{1}{1} \leq \|f\| \quad \text{so} \quad 1 \leq \|f\| \text{ ——— } (2)$$

from (1) we get

$$|f(x)| \leq \|x\|$$

so $$\frac{|f(x)|}{\|x\|} \leq 1 \quad \text{when } x \neq 0$$

$$\|f\| = \sup_{\substack{x \in \mathbb{C} \\ \|x\| \neq 0}} \left\{ \frac{|f(x)|}{\|x\|} \right\} \leq \sup 1 = 1$$

$$\|f\| \leq 1 \text{ ——— } (3)$$

from (2) and (3) we get that $\|f\| = 1$

also $z = (1, 0) \perp N(f)$

$$N(f) = \{x \in \mathbb{C}^2 , f(x) = 0\}$$

$x = (x_1, x_2)$

$f(x) = f(x_1, x_2) = 0$ when $(0, x_2)$

$\therefore f(0, x_2) = 0 \quad \forall x_2 \in \mathbb{C}$

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$\therefore N(f) = \{(0, x_2) / x_2 \in \mathbb{C}\}$

we to show that $z \perp N(f)$

as $z = (1, 0)$ Let $w = (0, w_2) \in N(f)$

$$\langle z / w \rangle = \langle (1,0), (0, w_2) \rangle$$
$$= 1 \cdot \bar{0} + 0 \, \overline{w_2}$$
$$= 0$$

$\therefore z \perp w \quad \forall w \in N(f)$

$\therefore z \perp N(f)$

Definition Sesquilinear form

Let X and Y be a vector spaces over the same field K. Then a sesquilinear form or sesquilinear functional) h on $X \times Y$ is a mapping $h : X \times Y \longrightarrow K$ such that for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ and all scalars $\alpha, \beta \in K$

a) $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$

b) $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$

c) $h(\alpha x, y) = \alpha h(x, y)$

d) $h(x, \beta y) = \bar{\beta} h(x, y)$

~~hence h is linear in the~~

Hence h is linear in the first argument and conjugate linear in the second one.

If X and Y are real then (d) is simply

$$h(x, \beta y) = \beta h(x, y)$$

and h is called bilinear since it is linear in both arguments

If X and Y are normed space and if there is a real number c such that for all x, y

$$|h(x,y)| \leq c\|x\| \, \|y\|$$

then h is said to be bounded and the ~~number~~

$$\|h\| = \sup_{\substack{0 \neq x \in X \\ 0 \neq y \in Y}} \frac{|h(x,y)|}{\|x\| \, \|y\|} = \sup_{\substack{\|x\| \leq 1 \\ \|y\| \leq 1}} |h(x,y)|$$

is called the norm of h.

also when h is bounded we write

$$|h(x,y)| \leq \|h\| \, \|x\| \, \|y\|$$