Surface Area
المساحة السطحية

Let f and its first derivatives are continuous on the closed region R in the xy −plane, then the area of the surface over R in given by:

$$S = \iint_R dS = \iint_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ \ dA$$

Or

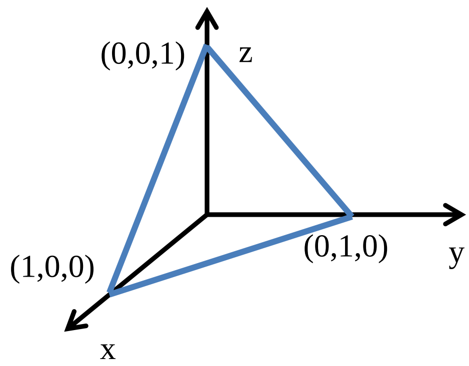
$$S = \iint_R \frac{\sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + [f_z(x,y)]^2}}{f_z(x,y)}\ \ dA$$

Example 1: Find the surface area $\iint xz\, dz$ for the plane $z = 1 - x - y$.

Solution:

$$dS = \sqrt{f_x{}^2 + f_y{}^2 + 1}\ dA$$

$$dS = \sqrt{(-1)^2 + (-1)^2 + 1}\ dA \Rightarrow dS = \sqrt{3}\ dA$$



$$S = \sqrt{3} \iint_R x(1 - x - y)dA$$

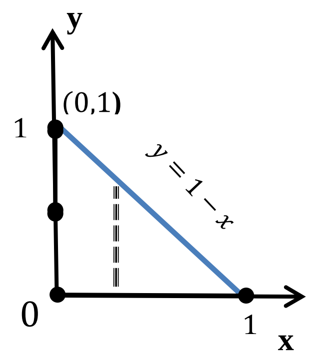
$$S = \sqrt{3} \int_0^1 \int_0^{1-x} (x - x^2 - xy)\, dy\, dx$$

$$S = \sqrt{3} \int_0^1 xy - x^2y - \frac{xy^2}{2}\Big]_0^{1-x}\ dx$$

$$S = \sqrt{3} \int_0^1 x(1-x) - x^2(1-x) - \frac{x(1-x)^2}{2}\ dx$$

$$S = \sqrt{3} \int_0^1 x - x^2 - x^2 + x^3 - \frac{x(1-x)^2}{2}\ dx$$

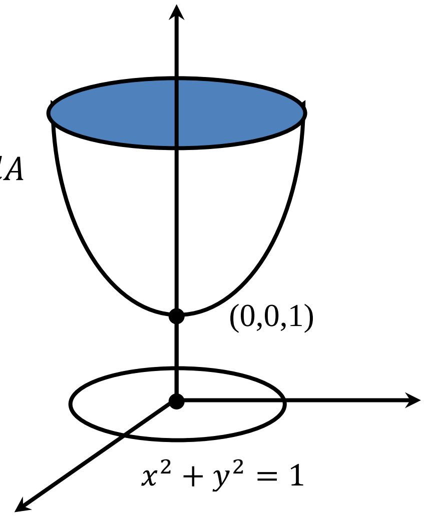
$$S = \sqrt{3}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x(1-x)^3}{6}\right]_0^1$$



Example 2: Find the surface area of paraboloid $z = 1 + x^2 + y^2$ that

lies above the unit circle i.e., $x^2 + y^2 = 1$.

Solution:



$$dS = \sqrt{f_x{}^2 + f_y{}^2 + 1}\ dA$$

$$dS = \sqrt{(2x)^2 + (2y)^2 + 1}\ dA \Rightarrow dS = \sqrt{4(x^2 + y^2) + 1}\ dA$$

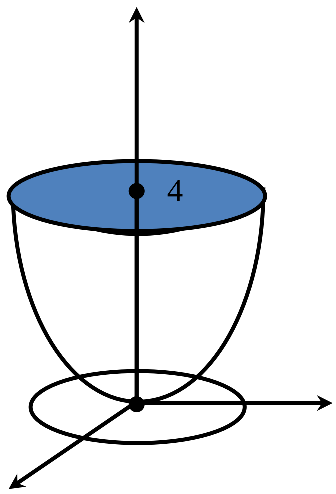
$$S = \iint_R \sqrt{4(x^2 + y^2) + 1}\, dA$$

$$S = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\ rdr\, d\theta$$

$$S = \frac{1}{8}\int_0^{2\pi} \frac{(4r^2 + 1)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^1 d\theta =$$

Example 3: Find the area of the surface cut from the bottom (الاسفل) of the paraboloid $x^2 + y^2 - z = 0$ by plane $z = 4$.

Solution:



$$dS = \sqrt{f_x{}^2 + f_y{}^2 + 1}\ dA$$

$$dS = \sqrt{(2x)^2 + (2y)^2 + 1}\ dA \Rightarrow$$

$$S = \iint_R \sqrt{4(x^2 + y^2) + 1}\, dA$$

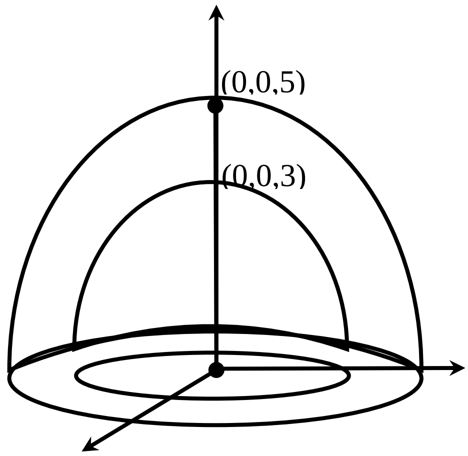
$$S = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\ rdr\, d\theta$$

$$S = \frac{1}{8}\int_0^{2\pi} \frac{(4r^2 + 1)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^2 d\theta =$$

Example 4: Find the surface area of the portion (الجزء) of the hemisphere

(نصف الكرة) $z = \sqrt{25 - x^2 - y^2}$ that lies above the region R bounded by the circle with radius 3 .

Solution:



$z = \sqrt{25 - x^2 - y^2} \;\; \Rightarrow \;\; x^2 + y^2 + z^2 = 25$

$$dS = \frac{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}{2z} \; dA$$

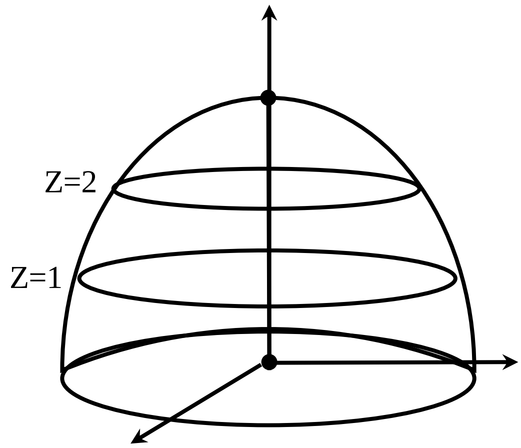
$$dS = \frac{5}{z} \; dA \Rightarrow \; dS = \frac{5}{\sqrt{25 - x^2 - y^2}} \; dA$$

$$S = \iint_R \frac{5}{\sqrt{25 - x^2 - y^2}} \, dA$$

$$S = \int_0^{2\pi} \int_3^5 \frac{5}{\sqrt{25 - r^2}} \, r dr \, d\theta \Rightarrow \;\; S = -\frac{5}{2} \int_0^{2\pi} \frac{(25 - r^2)^{\frac{1}{2}}}{\frac{1}{2}} \Big]_3^5 \, d\theta =$$

Example 5: Find the surface area of the portion (الجزء) of the sphere with radius 4 which lies above $z = 1$, below $z = 2$

Solution:



$x^2 + y^2 + z^2 = 16$

$$dS = \frac{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}{2z} \; dA$$

$$dS = \frac{4}{z} \; dA \Rightarrow \; dS = \frac{4}{\sqrt{16 - x^2 - y^2}} \; dA$$

$$S = \iint_R \frac{4}{\sqrt{16 - x^2 - y^2}} \, dA$$

$$S = \int_0^{2\pi} \int_{\sqrt{12}}^{\sqrt{15}} \frac{4}{\sqrt{16 - r^2}} \, r dr \, d\theta \Rightarrow$$