TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions :

1. (*i*) $\dfrac{720}{s^7}$ (*ii*) $\dfrac{40320}{s^9}$ (*iii*) $\dfrac{\Gamma\,(3/2)}{s^{3/2}}$ (*iv*) $\dfrac{1}{s-7}$

2. (*i*) $\dfrac{s}{s^2+25}$ (*ii*) $\dfrac{s}{s^2-49}$ (*iii*) $\dfrac{8}{s^2-64}$ (*iv*) $\dfrac{4}{s^2+16}$

3. (*i*) $\dfrac{60+6s^2+s^4}{s^7}$ (*ii*) $\dfrac{s-4}{s^2-16}$ (*iii*) $\dfrac{0.1s+0.9}{s^2+3.24}$ (*iv*) $\dfrac{s}{a^2s^2+b^2c^2}$

4. (*i*) $\dfrac{6}{s-4}+\dfrac{3}{s+5}+\dfrac{9}{s-7}$ (*ii*) $\dfrac{2}{2s+1}-\dfrac{4}{5s+7}+\dfrac{11}{1+5s}$

5. (*i*) $\dfrac{4}{s^2+1}-\dfrac{9s}{s^2+5}+\dfrac{6}{4s^2+1}+\dfrac{8s}{s^2+25}$ (*ii*) $\dfrac{2}{(s-1)(s+1)}+\dfrac{6s}{s^2-25}-\dfrac{4s}{s^2-9}+\dfrac{1}{6s^2-1}$

6. (*i*) $\dfrac{3}{s^2-1}-\dfrac{4}{s^2+9}+\dfrac{s}{s^2+16}-\dfrac{5}{s^2-16}$ (*ii*) $\dfrac{5}{3s^2+27}+\dfrac{6}{5s^2-9}-\dfrac{5}{36s^2+1}+\dfrac{9}{5+s^2}$.

Answers

1. (*i*) t^6 (*ii*) t^8 (*iii*) $\sqrt{t}$ (*iv*) e^{7t}

2. (*i*) $\cos 5t$ (*ii*) $\cosh 7t$ (*iii*) $\sinh 8t$ (*iv*) $\sin 4t$

3. (*i*) $\dfrac{1}{12}t^6+\dfrac{1}{4}t^4+\dfrac{1}{2}t^2$ (*ii*) e^{-4t}

(*iii*) $\dfrac{1}{10}\cos 1.8\,t+\dfrac{1}{2}\sin 1.8\,t$ (*iv*) $\dfrac{1}{a^2}\cos\dfrac{bc}{a}\,t$

4. (*i*) $6e^{4t}+3e^{-5t}+9e^{7t}$ (*ii*) $e^{-\frac{1}{2}t}-\dfrac{4}{5}e^{-\frac{7}{5}t}+\dfrac{11}{5}e^{-\frac{1}{5}t}$

5. (*i*) $4\sin t-9\cos\sqrt{5}\,t+3\sin\dfrac{t}{2}+8\cos 5t$ (*ii*) $2\sinh t+6\cosh 5t-4\cosh 3t+\dfrac{1}{\sqrt{6}}\sinh\dfrac{t}{\sqrt{6}}$

6. (*i*) $3\sinh t-\dfrac{4}{3}\sin 3t+\cos 4t-\dfrac{5}{4}\sinh 4t$ (*ii*) $\dfrac{5}{9}\sin 3t+\dfrac{2}{\sqrt{5}}\sinh\dfrac{3}{\sqrt{5}}t-\dfrac{5}{6}\sin\dfrac{t}{6}+\dfrac{9}{\sqrt{5}}\sin\sqrt{5}t.$

2.6. VALUE OF $L^{-1}(F(s-a))$ IN TERMS OF $L^{-1}(F(s))$

By the **first shifting theorem** of Laplace transforms,

if $\qquad L(f(t)) = F(s)$, then $L(e^{at} f(t)) = F(s-a)$.

$\therefore$ If $\quad$ **$L^{-1}(F(s)) = f(t)$ then $L^{-1}(F(s-a)) = e^{at} f(t)$.**

Equivalently, $L^{-1}(F(s-a)) = e^{at}\, L^{-1}(F(s))$.

Remark. In the formula, $L^{-1}(F(s-a)) = e^{at}\,L^{-1}(F(s))$, if we replace a by $-a$, the formula takes the form,

$$\mathbf{L^{-1}(F(s+a)) = e^{-at}\, L^{-1}(F(s))}.$$

For example, $\quad L^{-1}\left(\dfrac{s-2}{(s-2)^2+5^2}\right) = e^{2t}\,L^{-1}\left(\dfrac{s}{s^2+5^2}\right) = e^{2t}\cos 5t.$

Theorem. Using the formula $L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s))$ prove that :

1. $\mathbf{L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}}$

2. $\mathbf{L^{-1}\left(\dfrac{\Gamma(b+1)}{(s-a)^{b+1}}\right) = e^{at}\, t^{b}\,,\ b > -1}$

3. $\mathbf{L^{-1}\left(\dfrac{n!}{(s-a)^{n+1}}\right) = e^{at}\, t^{n}\,,\ n = 0, 1, 2, \ldots\ldots}$

4. $\mathbf{L^{-1}\left(\dfrac{b}{(s-a)^2 - b^2}\right) = e^{at} \sinh bt}$

5. $\mathbf{L^{-1}\left(\dfrac{s-a}{(s-a)^2 - b^2}\right) = e^{at} \cosh bt}$

6. $\mathbf{L^{-1}\left(\dfrac{b}{(s-a)^2 + b^2}\right) = e^{at} \sin bt}$

7. $\mathbf{L^{-1}\left(\dfrac{s-a}{(s-a)^2 + b^2}\right) = e^{at} \cos bt.}$

Proof. 1. We have $\quad L^{-1}\left(\dfrac{1}{s}\right) = 1.$

$$\therefore \qquad L^{-1}\left(\frac{1}{s-a}\right) = e^{at}\,.\,L^{-1}\left(\frac{1}{s}\right) = e^{at}\,.\,1$$

$$\therefore \qquad \mathbf{L^{-1}\left(\frac{1}{s-a}\right) = e^{at}}.$$

2. We have $\quad L^{-1}\left(\dfrac{\Gamma(b+1)}{s^{b+1}}\right) = t^{b}\,,\, b > -1.$

$$\therefore \qquad L^{-1}\left(\frac{\Gamma(b+1)}{(s-a)^{b+1}}\right) = e^{at}\, L^{-1}\left(\frac{\Gamma(b+1)}{s^{b+1}}\right) = e^{at}\, t^{b}$$

$$\therefore \qquad \mathbf{L^{-1}\left(\frac{\Gamma(b+1)}{(s-a)^{b+1}}\right) = e^{at}\, t^{b}\,,\ b > -1}.$$

3. We have $\quad L^{-1}\left(\dfrac{n!}{s^{n+1}}\right) = t^{n}\,,\ n = 0, 1, 2, \ldots\ldots$

$$\therefore \qquad L^{-1}\left(\frac{n!}{(s-a)^{n+1}}\right) = e^{at}\, L^{-1}\left(\frac{n!}{s^{n+1}}\right) = e^{at}\, t^{n}$$

$$\therefore \qquad \mathbf{L^{-1}\left(\frac{n!}{(s-a)^{n+1}}\right) = e^{at}\, t^{n}\,,\ n = 0, 1, 2, \ldots\ldots}$$

4. We have $\mathrm{L}^{-1}\left(\dfrac{b}{s^2-b^2}\right) = \sinh bt$.

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{b}{(s-a)^2-b^2}\right) = e^{at}\,\mathrm{L}^{-1}\left(\frac{b}{s^2-b^2}\right) = e^{at}\sinh bt$$

$$\therefore \quad \mathbf{L^{-1}\left(\frac{b}{(s-a)^2-b^2}\right) = e^{at}\sinh bt}.$$

5. We have $\mathrm{L}^{-1}\left(\dfrac{s}{s^2-b^2}\right) = \cosh bt$.

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{s-a}{(s-a)^2-b^2}\right) = e^{at}\,\mathrm{L}^{-1}\left(\frac{s}{s^2-b^2}\right) = e^{at}\cosh bt$$

$$\therefore \quad \mathbf{L^{-1}\left(\frac{s-a}{(s-a)^2-b^2}\right) = e^{at}\cosh bt}.$$

6. We have $\mathrm{L}^{-1}\left(\dfrac{b}{s^2+b^2}\right) = \sin bt$.

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\,\mathrm{L}^{-1}\left(\frac{b}{s^2+b^2}\right) = e^{at}\sin bt$$

$$\therefore \quad \mathbf{L^{-1}\left(\frac{b}{(s-a)^2+b^2}\right) = e^{at}\sin bt}.$$

7. We have $\mathrm{L}^{-1}\left(\dfrac{s}{s^2+b^2}\right) = \cos bt$.

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\,\mathrm{L}^{-1}\left(\frac{s}{s^2+b^2}\right) = e^{at}\cos bt$$

$$\therefore \quad \mathbf{L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right) = e^{at}\cos bt.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the function* $\dfrac{12}{(s-4)^3}$.

Sol.
$$\mathrm{L}^{-1}\left(\frac{12}{(s-4)^3}\right) = 6\mathrm{L}^{-1}\left(\frac{2}{(s-4)^3}\right)$$

$$= 6e^{4t}\,\mathrm{L}^{-1}\left(\frac{2}{s^3}\right) = 6e^{4t}\,\mathrm{L}^{-1}\left(\frac{2!}{s^{2+1}}\right)$$

$(\because\ \mathrm{L}^{-1}(\mathrm{F}(s-a)) = e^{at}\,\mathrm{L}^{-1}(\mathrm{F}(s)))$

$$= \mathbf{6e^{4t}\,t^2}.$$

Example 2. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{3s+1}{(s+1)^4}$ (*ii*) $\displaystyle\sum_{k=1}^{5} \frac{\lambda_k}{s+k^2}$.

Sol. (*i*) $$L^{-1}\left(\frac{3s+1}{(s+1)^4}\right) = L^{-1}\left(\frac{3(s+1)-2}{(s+1)^4}\right)$$

$$= e^{-t}\, L^{-1}\left(\frac{3s-2}{s^4}\right) = e^{-t}\, L^{-1}\left(\frac{3}{s^3} - \frac{2}{s^4}\right)$$

($\because$ $L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s))$)

$$= e^{-t}\left[\frac{3}{2} L^{-1}\left(\frac{2!}{s^3}\right) - \frac{2}{6} L^{-1}\left(\frac{3!}{s^4}\right)\right] = \mathbf{e^{-t}\left[\frac{3}{2}t^2 - \frac{1}{3}t^3\right]}.$$

(*ii*) $$L^{-1}\left(\sum_{k=1}^{5} \frac{\lambda_k}{s+k^2}\right) = \sum_{k=1}^{5} \lambda_k\, L^{-1}\left(\frac{1}{s+k^2}\right) = \sum_{\mathbf{k=1}}^{\mathbf{5}} \boldsymbol{\lambda_k\, e^{-k^2 t}}.$$

Example 3. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{s+3}{(s+3)^2+4}$ (*ii*) $\dfrac{3s}{s^2+2s-8}$.

Sol. (*i*) $$L\left(\frac{s+3}{(s+3)^2+4}\right) = e^{-3t}\, L^{-1}\left(\frac{s}{s^2+2^2}\right) = \mathbf{e^{-3t}\cos 2t}.$$

($\because$ $L^{-1}(F(s-a)) = e^{at} L^{-1}(F(s))$)

(*ii*) $$\frac{3s}{s^2+2s-8} = \frac{3s}{(s+1)^2-9} = \frac{3s+3-3}{(s+1)^2-9} = \frac{3(s+1)}{(s+1)^2-9} - \frac{3}{(s+1)^2-9}$$

$$\therefore \quad L^{-1}\left(\frac{3s}{s^2+2s-8}\right) = L^{-1}\left(\frac{3(s+1)}{(s+1)^2-9} - \frac{3}{(s+1)^2-9}\right)$$

$$= 3L^{-1}\left(\frac{s+1}{(s+1)^2-9}\right) - L^{-1}\left(\frac{3}{(s+1)^2-9}\right)$$

$$= 3e^{-1.t}\, L^{-1}\left(\frac{s}{s^2-3^2}\right) - e^{-1.t}\, L^{-1}\left(\frac{3}{s^2-3^2}\right)$$

$$= 3e^{-t}\cosh 3t - e^{-t}\sinh 3t$$

$$= \mathbf{e^{-t}(3\cosh 3t - \sinh 3t)}.$$

Example 4. *Find the inverse Laplace transform of the function*

$$\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}.$$

Sol. $$\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}$$

$$= \frac{1}{(s+2)^2+9} - \frac{s+4}{(s+4)^2+81} + \frac{s+2}{(s-2)^2+25}$$

$$= \frac{1}{(s+2)^2+9} - \frac{s+4}{(s+4)^2+81} + \frac{s-2}{(s-2)^2+25} + \frac{4}{(s-2)^2+25}$$

$$\therefore \quad L^{-1}\left(\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}\right)$$

$$= L^{-1}\left(\frac{1}{(s+2)^2+9}\right) - L^{-1}\left(\frac{s+4}{(s+4)^2+81}\right) + L^{-1}\left(\frac{s-2}{(s-2)^2+25}\right) + L^{-1}\left(\frac{4}{(s-2)^2+25}\right)$$

$$= e^{-2t} L^{-1}\left(\frac{1}{s^2+9}\right) - e^{-4t} L^{-1}\left(\frac{s}{s^2+81}\right) + e^{2t} L^{-1}\left(\frac{s}{s^2+25}\right) + e^{2t} L^{-1}\left(\frac{4}{s^2+25}\right)$$

$$= \frac{1}{3} e^{-2t} L^{-1}\left(\frac{3}{s^2+3^2}\right) - e^{-4t} L^{-1}\left(\frac{s}{s^2+9^2}\right) + e^{2t} L^{-1}\left(\frac{s}{s^2+5^2}\right) + \frac{4}{5} e^{2t} L^{-1}\left(\frac{5}{s^2+5^2}\right)$$

$$= \mathbf{\frac{1}{3} e^{-2t} \sin 3t - e^{-4t} \cos 9t + e^{2t} \cos 5t + \frac{4}{5} e^{2t} \sin 5t.}$$

2.7. VALUE OF L^{-1} (e^{-as} F(s)) IN TERMS OF L^{-1} (F(s))

By the **second shifting theorem** of Laplace transforms,

if $\quad L(f(t)) = F(s)$ then for any $a \geq 0$, $L(f(t-a)\, u_a(t)) = e^{-as} F(s)$.

$\therefore$ **If $L^{-1}(F(s)) = f(t)$ then $L^{-1}(e^{-as} F(s)) = f(t-a)\, u_a(t)$ for any $a \geq 0$.**

Remark. It may be recalled that for $a \geq 0$, $u_a(t)$ is the *unit step function* defined as :

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a. \end{cases}$$

$\therefore$ The function $f(t-a)\, u_a(t)$ can also be written as

$$f(t-a)\, u_a(t) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a. \end{cases}$$

Example 5. *Find the inverse Laplace transform of the function* $\frac{2s+1}{s^2 e^{2s}}$.

Sol. $\quad \frac{2s+1}{s^2 e^{2s}} = e^{-2s} \cdot \frac{2s+1}{s^2} = e^{-2s} F(s)$, where $F(s) = \frac{2s+1}{s^2}$.

$$L^{-1}(F(s)) = L^{-1}\left(\frac{2s+1}{s^2}\right) = L^{-1}\left(\frac{2}{s} + \frac{1}{s^2}\right) = 2L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right)$$

$= 2(1) + t = t + 2 = f(t)$, say.

$$\therefore \quad L^{-1}\left(\frac{2s+1}{s^2 e^{2s}}\right) = L^{-1}(e^{-2s}F(s))$$

$$= f(t-2)\, u_2(t) = ((t-2)+2)\, u_2(t) = t \cdot u_2(t) = \begin{cases} 0 & \text{if } t < 2 \\ t & \text{if } t > 2 \end{cases}.$$

$$\therefore \quad L^{-1}\left(\frac{2s+1}{s^2 e^{2s}}\right) = g(t), \text{ where } g(t) = \begin{cases} 0 & \text{if } t < 2 \\ t & \text{if } t > 2 \end{cases}.$$

Example 6. *Find the inverse Laplace transform of the function* $\dfrac{e^{-\pi s/2}\, s}{s^2+1}$.

Sol. $$\frac{e^{-\pi s/2} s}{s^2+1} = e^{-(\pi/2)s} \cdot \frac{s}{s^2+1} = e^{-(\pi/2)s}\, F(s), \text{ where } F(s) = \frac{s}{s^2+1}.$$

$$L^{-1}(F(s) = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t = f(t), \text{ say.}$$

$$\therefore \quad L^{-1}\left(\frac{e^{-\pi s/2}\, s}{s^2+1}\right) = L^{-1}\left(e^{-(\pi/2)s}\, F(s)\right)$$

$$= f\left(t-\frac{\pi}{2}\right) u_{\pi/2}(t) = \cos\left(t-\frac{\pi}{2}\right) u_{\pi/2}(t)$$

$$= \sin t \,.\, u_{\pi/2}(t) = \begin{cases} \mathbf{0} & \textbf{if } \mathbf{t < \pi/2} \\ \mathbf{\sin t} & \textbf{if } \mathbf{t > \pi/2} \end{cases}.$$

Example 7. *Find the inverse Laplace transform of the function* $\dfrac{3\,(1-e^{-\pi s})}{s^2+9}$.

Sol. $$L^{-1}\left(\frac{3(1-e^{-\pi s})}{s^2+9}\right) = L^{-1}\left(\frac{3}{s^2+9} - \frac{3e^{-\pi s}}{s^2+9}\right) = L^{-1}\left(\frac{3}{s^2+9}\right) - L^{-1}\left(\frac{3e^{-\pi s}}{s^2+9}\right)$$

$$\therefore \quad L^{-1}\left(\frac{3(1-e^{-\pi s})}{s^2+9}\right) = \sin 3t - L^{-1}\left(\frac{3e^{-\pi s}}{s^2+9}\right) \qquad \ldots(1)$$

$$\frac{3e^{-\pi s}}{s^2+9} = e^{-\pi s} \cdot \frac{3}{s^2+9} = e^{-\pi s}\, F(s), \text{ where } F(s) = \frac{3}{s^2+9}.$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{3}{s^2+9}\right) = \sin 3t = f(t), \text{ say.}$$

$$\therefore \quad L^{-1}\left(\frac{3e^{-\pi s}}{s^2+9}\right) = L^{-1}(e^{-\pi s}\, F(s)) = f(t-\pi)\, u_\pi(t) = \sin 3(t-\pi)\, u_\pi(t)$$

$$= -\sin(3\pi - 3t)\, u_\pi(t) = -\sin 3t \,.\, u_\pi(t).$$

$$\therefore \quad (1) \Rightarrow L^{-1}\left(\frac{3(1-e^{-\pi s})}{s^2+9}\right) = \sin 3t - (-\sin 3t \,.\, u_\pi(t))$$

$$= \mathbf{(1 + u_\pi(t)) \sin 3t.}$$

2.8. VALUE OF L^{-1} F(s/a)) IN TERMS OF L^{-1} (F(s))

By the **change of scale property,**

if $$L(f(t)) = F(s) \text{ then for any } a > 0,\ L(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right) \quad \textit{i.e.} \quad L(af(at)) = F\left(\frac{s}{a}\right).$$

$\therefore$ **If $\mathbf{L^{-1}}$ (F(s)) = f(t) then** $\mathbf{L^{-1}\left(F\left(\frac{s}{a}\right)\right)}$ **= a f(at) for any a > 0.**

Example 8. *Find the inverse Laplace transform of the following functions :*

(*i*) $\dfrac{a}{\left(\dfrac{s}{\lambda}\right)^2 + a^2}$ (*ii*) $\dfrac{s}{\left(\dfrac{s}{7}\right)^2 + 9}$

Sol. (*i*) Let $\quad F(s) = \dfrac{a}{s^2 + a^2}$.

$\therefore \quad L^{-1}(F(s)) = \sin at = f(t), \quad$ say

$$\therefore \quad L^{-1}\left(\frac{a}{\left(\frac{s}{\lambda}\right)^2 + a^2}\right) = L^{-1}\left(F\left(\frac{s}{\lambda}\right)\right) = \lambda f(\lambda t) = \boldsymbol{\lambda \sin \lambda at.}$$

Alternative method

$$L^{-1}\left(\frac{a}{\left(\frac{s}{\lambda}\right)^2 + a^2}\right) = L^{-1}\left(\frac{\lambda^2 a}{s^2 + a^2\lambda^2}\right) = \lambda L^{-1}\left(\frac{\lambda a}{s^2 + (\lambda a)^2}\right) = \boldsymbol{\lambda \sin \lambda at.}$$

(*ii*) Let $\quad F(s) = \dfrac{s}{s^2 + 9}$.

$\therefore \quad L^{-1}(F(s)) = \cos 3t = f(t), \quad$ say

$$\therefore \quad L^{-1}\left(\frac{s}{\left(\frac{s}{7}\right)^2 + 9}\right) = 7L^{-1}\left(\frac{s/7}{(s/7)^2 + 9}\right) = 7(7\, f(7t))$$

$$= 49 \cos(3(7t)) = \mathbf{49 \cos 21t.}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}(F(s - a)) = e^{at} f(t)$.

Rule II. If $L^{-1}(F(s)) = f(t)$, then

$L^{-1}(e^{-as} F(s)) = f(t - a)\, u_a(t),\ a \geq 0.$

Rule III. If $L^{-1}(F(s)) = f(t)$, then

$$L^{-1}\left(F\left(\frac{s}{a}\right)\right) = a\, f(at),\ a > 0.$$

TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions :

1. (i) $\dfrac{1}{(s-4)^5}$ (ii) $\dfrac{4}{(s+2)^7}$

2. (i) $\dfrac{1}{s^2+6s+13}$ (ii) $\dfrac{s+3}{s^2-4s+13}$

3. (i) $\dfrac{s}{\left(s+\frac{1}{2}\right)^2+1}$ (ii) $\dfrac{4}{s^2+6s+18}$

4. (i) $\dfrac{5}{s^2-4s-3}$ (ii) $\dfrac{9}{s^2+s+\frac{1}{2}}$

5. (i) $\dfrac{s+2}{s^2+4s+7}$ (ii) $\dfrac{s+10}{s^2+8s+20}$

6. (i) $\dfrac{1}{(s-4)^5}+\dfrac{5}{s^2-4s+29}$ (ii) $\dfrac{1}{(s-3)^4}+\dfrac{s+3}{s^2+6s+45}$

7. (i) $\dfrac{1}{2s-5}+\dfrac{5s-2}{3s^2+s+8}+\dfrac{s+1}{s^2+2s+2}$

(ii) $\dfrac{1}{s^2-6s+10}+\dfrac{1}{s^2+8s+16}+\dfrac{3s-2}{s^2-4s+20}+\dfrac{3s+7}{s^2-2s-3}$

8. (i) $\dfrac{e^{-s}}{s^2}$ (ii) $\dfrac{e^{-3s}}{s^3}$

9. (i) $\dfrac{(5s+1)\,e^{-5s}}{s^2}$ (ii) $\dfrac{2+2s+s^2}{s^3\,e^s}$

10. (i) $-\dfrac{e^{-\pi s/2}}{s^2+1}$ (ii) $-\dfrac{5se^{-\pi s}}{s^2+1}$

11. (i) $\dfrac{se^{-2s}}{s^2+\pi^2}$ (ii) $\dfrac{e^{-2\pi s/3}}{s^2+1}$

12. (i) $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$ (ii) $\dfrac{(s+1)\,e^{-\pi s}}{s^2+s+1}$.

Answers

1. (i) $\dfrac{1}{24}t^4\,e^{4t}$ (ii) $\dfrac{1}{180}t^6\,e^{-2t}$

2. (i) $\dfrac{1}{2}e^{-3t}\sin 2t$ (ii) $e^{2t}\left(\cos 3t+\dfrac{5}{3}\sin 3t\right)$

3. (i) $e^{-t/2}\left(\cos t-\dfrac{1}{2}\sin t\right)$ (ii) $\dfrac{4}{3}e^{-3t}\sin 3t$

4. (i) $\dfrac{5}{\sqrt{7}}e^{2t}\sinh\sqrt{7}t$ (ii) $18e^{-t/2}\sin\dfrac{t}{2}$

5. (i) $e^{-2t}\cos\sqrt{3}t$ (ii) $e^{-4t}\,[\cos 2t+3\sin 2t]$

6. (*i*) $\frac{1}{24} e^{4t} t^4 + e^{2t} \sin 5t$ (*ii*) $\frac{1}{6} e^{3t} t^3 + e^{-3t} \cos 6t$

7. (*i*) $\frac{1}{2} e^{5t/2} + e^{-\frac{t}{6}} \left[\frac{5}{3} \cos \frac{\sqrt{95}}{6} t - \frac{25}{3\sqrt{95}} \sin \frac{\sqrt{95}}{6} t \right] + e^{-t} \cos t$

(*ii*) $e^{3t} \sin t + te^{-4t} + 3e^{2t} \cos 4t + e^{2t} \sin 4t + 3e^t \cosh 2t + 5e^t \sinh 2t$

8. (*i*) $(t-1)\, u_1(t)$ (*ii*) $\frac{1}{2} (t-3)^2\, u_3(t)$

9. (*i*) $t \,.\, u_5(t)$ (*ii*) $t^2 u_1(t)$

10. (*i*) $\cos t \,.\, u_{\pi/2}(t)$ (*ii*) $5 \cos t \,.\, u_\pi(t)$

11. (*i*) $\cos \pi t \,.\, u_2(t)$ (*ii*) $-\cos\left(\frac{\pi}{6} - t\right) . u_{2\pi/3}(t)$

12. (*i*) $\frac{4}{3\sqrt{\pi}} e^{16-4t} (t-3)^{3/2} u_3(t)$

(*ii*) $e^{-\frac{1}{2}(t-\pi)} \left[\cos \frac{\sqrt{3}}{2} (t-\pi) + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} (t-\pi) \right] u_\pi(t).$

2.9. VALUE OF L⁻¹ (F(s)/s) IN TERMS OF L⁻¹ (F(s))

By the property of **Laplace transforms of integrals,**

if $$\mathrm{L}(f(t)) = \mathrm{F}(s), \text{ then } \mathrm{L}\left(\int_0^t f(\mathrm{T})\, d\mathrm{T} \right) = \frac{1}{s} \mathrm{F}(s).$$

$\therefore$ If $$\mathbf{L^{-1}\,(F(s)) = f(t), \text{ then } L^{-1}\left(\frac{1}{s} F(s)\right) = \int_0^t f(T)\, dT.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the inverse Laplace transform of the function* $\frac{1}{s(s^2 + a^2)}$.

Sol. $$\frac{1}{s(s^2+a^2)} = \frac{1}{s}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{s} \mathrm{F}(s), \text{ where } \mathrm{F}(s) = \frac{1}{s^2+a^2}.$$

$$\mathrm{L}^{-1}(\mathrm{F}(s)) = \mathrm{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \mathrm{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a} \sin at = f(t), \text{ say.}$$

$\therefore$ By the **property of Laplace transform of integrals,**

$$\mathrm{L}^{-1}\left(\frac{1}{s} \mathrm{F}(s)\right) = \int_0^t f(\mathrm{T})\, d\mathrm{T} = \int_0^t \frac{1}{a} \sin a\mathrm{T}\, d\mathrm{T} = \frac{1}{a}\left(-\frac{\cos a\mathrm{T}}{a}\right)\Bigg|_0^t$$

$$= -\frac{1}{a^2}(\cos at - \cos 0) = \frac{1 - \cos at}{a^2}$$

$$\therefore \quad \mathrm{L}^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = \mathbf{\frac{1 - \cos at}{a^2}}.$$