Karnaugh maps

A Karnaugh map (also called a K-map) is a version of a truth table for a Boolean expression that is laid out in a way that makes it easier to simplify.

Karnaugh Map and Truth Table

Take the expression $A \vee (A \wedge B)$.

The truth table for this expression looks like this:

A	B	$A \vee (A \wedge B)$
0	0	0
0	1	0
1	0	1
1	1	1

The truth table has four rows because there are two inputs and therefore 2^2 combinations of 1s and 0s. Each row shows the output for each combination of input values.

A Karnaugh map is a variation on a truth table, which is arranged such that:

- The number of output cells matches the number of rows in the truth table
- The possible input values for the variables (in the expression) form the row and column headings
- The output for each of combination of input values is written at the row/column intersections

In **Figure 1** below, you can see that the inputs are A and B. The possible values for A have been used as column headings and the possible values for B have been used as row headings. At the intersection of the row and column is the output from the circuit for the given inputs.

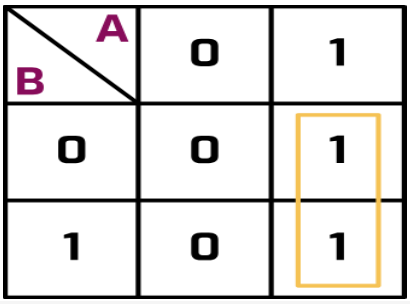

Figure 1: A two-input Karnaugh map

Looking at this Karnaugh map, you can see that the there is a group of 1s highlighted. In a Karnaugh map, you circle any groups that have a bank of 1s that is a power of 2 (ie when there are 1, 2, 4, 8 cells in the group, but not when there are 3, 5, 6 etc). Here, there is only one group and it is in the column that shows the outputs where $A = 1$. Because this is the only circled group, you know the equation simplifies to A.

Drawing the Karnaugh map:

Now examine the steps of drawing a Karnaugh map in more detail. Consider once again the expression $A \vee (A \wedge B)$.

To build the map, you follow these steps:

1. Create a grid where the number of rows and columns is determined by the number of variables in the expression
2. Break up the equation at the ORs; this will give you two parts: A and $A \wedge B$
3. Consider the first part: write a 1 wherever A is True, meaning you write 1s in the cells in the right-hand column
4. Consider the second part: write a 1 wherever $A \wedge B$ is True; this would mean writing a 1 in the cell in the bottom right-hand corner, but there is already a 1 there, so you don't need to do anything
5. Write 0s in the empty cells (or you can just leave these cells blank)
6. Circle any groups that have that have a number of 1s that is a power of 2 (eg groups of 1, 2, 4, 8 cells)

A summary of the rules

- Groups must be horizontal or vertical, no diagonals are allowed
- The number of cells in each group must be a power of 2
- Groups should be as large as possible
- Every 1 in the map must be in at least one group
- You must create the smallest number of groups possible

Examples

Now look at the expression $A \wedge B \vee \neg A \wedge B$.

1. Create a grid with two rows and two columns.
2. Break up the equation at the ORs; you get two separate expressions: $A \wedge B$ and $\neg A \wedge B$.
3. Put a 1 in the table for outputs where the expression $A \wedge B$ is True (**Figure 2**).

B \ A	0	1
0		
1		1

Figure 2: $A \land B$

4. Put a 1 in the table for outputs where the expression $\neg A \land B$ is True (**Figure 3**). (There is already a 1 in the bottom right-hand cell so ignore this.)

B \ A	0	1
0		
1	1	1

Figure 3: $\neg A \land B$

5. If you like, write 0s in the blank spaces.
6. Now circle any groups that have a number of 1s that is a power of 2 (eg groups of 1, 2, 4, 8 cells) (**Figure 4**).

B \ A	0	1
0	0	0
1	1	1

Figure 4: Circle the group of two 1s

The circled group shows that the output is 1 when B is 1, so this expression simplifies to B.

Let's look at a more complex Karnaugh map with more than one group of 1s and derive the expression it represents.

For each circled group in **Figure 5**, look to see which input(s) it relates to.

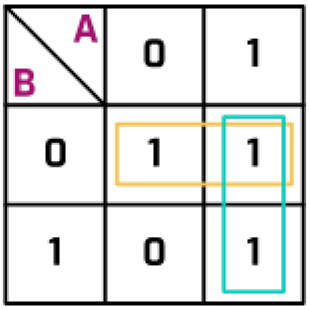

Figure 5: Two overlapping groups of 1s

The yellow, horizontal group shows that the output is always 1 when B is 0. This means the group represents $\neg B$.

The turquoise, vertical group shows that the output is always 1 when A is 1. This means the group represents A.

Since you can join the expressions for the two groups together using ∨, this map represents the expression $A \vee \neg B$

For a Karnaugh map with three inputs, one row or column is used to represent two of the inputs.

Consider the expression:

$(A \wedge B) \vee (A \wedge B \wedge C)$

The map for this expression is shown below in **Figure 6**. In this map:

- There are four rows to represent the possible values of the inputs AB
- There are two columns to represent the possible values of the input C

It does not matter how you split the inputs, so the map in this example could have been drawn with two rows and four columns. However, **the order of the input values is important**: in a row or column with two inputs (in this case AB), only one value of the pair of input values is allowed to change from one cell to the next. This means that you cannot have $0\ 1$ in one cell and $1\ 0$ in the next. This means that the order of rows is different from the usual order of rows in a truth table.

C / AB	0	1
00		
01		
11		
10		

Figure 6: A three-input Karnaugh map

Now the map can be populated:

- Split the expression into two expressions where there is an ∨, so you get $A \wedge B$ and $A \wedge B \wedge C$
- Put a 1 in the cells where $A \wedge B$ is True
- Put a 1 in the cells where $A \wedge B \wedge C$ is True (in this case, the 1 has already been written)
- Circle groups containing a number of 1s that is a power of 2 (eg groups of 1, 2, 4, 8 cells)

In **Figure 7** below, you can see the completed map in two different forms. The map on the right has the inputs grouped differently, but shows the same result, demonstrating that the grouping does not matter.

AB \ C	0	1
00	0	0
01	0	0
11	1	1
10	0	0

BC \ A	0	1
00	0	0
01	0	0
11	0	1
10	0	1

Figure 7: Input grouping does not matter

The circled group of 1s represents $A \wedge B$ (the output is 1 when both A and B are 1), meaning $A \wedge B$ is the simplified expression.

- Karnaugh maps in A level exam questions can contain up to four inputs. Let's look at the Karnaugh map for the equation $C \vee (C \wedge A \wedge B \wedge D)$.

The empty map for this expression is shown below in **Figure 8**. In this map:

- There are four columns to represent the possible values of the inputs AB
- There are four rows to represent the possible values of the input CD
- Only one value of each pair of input values changes from one cell to the next

AB / CD	00	01	11	10
00				
01				
11				
10				

Figure 8: A four-input Karnaugh map

Using the same method as before, 1s are placed on the map for each of the the two parts of the equation:

- C
- $C \wedge A \wedge B \wedge D$

AB / CD	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	1	1	1	1
10	1	1	1	1

Figure 9: Circle the group of eight 1s

This time (**Figure 9**) there is a big group of eight 1s to circle. This shows that the output is always 1 when C is 1, meaning the expression can be simplified to C.

- Now look at another four-input Karnaugh map — one with two overlapping groups of 1s (**Figure 10**)

CD \ AB	00	01	11	10
00	1	0	0	0
01	1	0	0	0
11	1	1	1	1
10	1	1	1	1

Figure 10: Two overlapping groups

You already know from the previous section that the large group of eight 1s represents C. If you examine which inputs do not change for the vertical group of 1s, you see that A and B are both always 0 for this group. This means the group is represented by $\neg A \wedge \neg B$, which you can simplify using De Morgan's laws to $\neg(A \vee B)$

Join the two expressions together with an ∨ to get the expression represented by this Karnaugh map: $\neg(A \vee B) \vee C$.

- In the four-input map in **Figure 11**, the two groups of 1s (circled yellow) are actually a single group of four 1s that overlaps the edge and wraps around.

CD \ AB	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	1	1	1	1
10	1	1	1	1

Figure 11: A group overlapping the map's edge

You can prove this is correct if you look at the inputs: for all of the 1s in this group, B is always 0 and C is always 1. So this group is represented by $C \wedge \neg B$

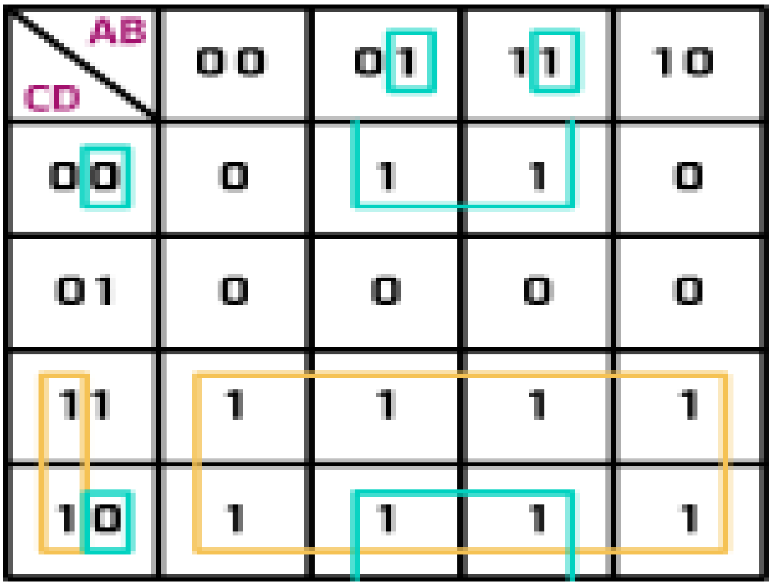

Figure 12: Two overlapping groups

The Karnaugh map in **Figure 12** has two groups that overlap, and one also overlaps the edge. Can you work out the expression that the map represents?