2.2 Bounded and continuous Linear Operators. (1)

Bounded ~~linear~~ operator.

Definition $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$

Let X and Y be two normed spaces and $T : X \to Y$ ~~[illegible]~~ operator. Then The operator T is said to be bounded if there is a real number c such that for all $x \in X$

$$\|T(x)\|_Y \leq c\|x\|_X \qquad \forall x \in X \quad \text{—(1)}$$

we must know that the left inquality is the norm in normed space $(Y, \|\cdot\|_Y)$ and the right norm in normed space $\|\cdot\|_X$, in general we use just the symbol $\|\cdot\|$, to denoted the norm of element. The element which belong to each normed space is deferment the norm it will be used.

2 Note The smallest possible c such that

1 is still holds for a non-zero $x \in X$ is

can written as $\frac{\|T(x)\|}{\|x\|} < c$

so the latest upper bound for c is

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{\|T(x)\|}{\|x\|} \right\}$$

$\|T\|$ is called the norm of the operator T.

If $X = \{0\}$ we define

$\|T\| = 0$

The inequality 1 it well be write as

$$\|T(x)\| \leq \|T\| \, \|x\| \quad \forall x \in X$$

Example

1 for Identity operator

The identity operator $I : X \longrightarrow X$ where X is a non-zero normed space. is a bounded and has norm $\| I \| = 1$

<u>Solution</u> since $I(x) = x \quad \forall x \in X$, we have

$$\| I(x) \| = \| x \| \qquad \forall x \in X$$

so the constant $c = 1$

$$\| I \| = \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{\| I(x) \|}{\| x \|} \right\} = \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{\| x \|}{\| x \|} \right\}$$

$$= \sup_{\substack{x \in X \\ x \neq 0}} 1 = 1$$

$\therefore \| I \| = 1$

2 Zero operator

The zero operator $O : X \longrightarrow X$ on a normed space X is a bonded operator and has norm $\| O \| = 0$

(4) Solution

$O(x) = \theta \quad \forall x \in X$ from definition of zero operator

So

$\|O(x)\| = \|\theta\| = 0 \leq \|x\| \quad \forall x \in X$

So O is bounded

$$\|O\| = \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{\|O(x)\|}{\|x\|} \right\} = \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{\|\theta\|}{\|x\|} \right\}$$

$$= \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{0}{\|x\|} \right\} = \sup_{\substack{x \in X \\ x \neq 0}} 0 = 0$$

Lemma (norm)

Let T be a bounded linear operator an alternative formula for the norm of T is $\|T\| = \sup\limits_{\substack{x \in X \\ \|x\| = 1}} \{ \|T(x)\| \}$

Proof: Let $\|x\| = a$ for $x \in X$ and set ~~By considering~~ $y = (\frac{1}{a}) x$, Then

$$\|y\| = \|(\tfrac{1}{a}) x\| =$$

$$= |\tfrac{1}{a}| \|x\|$$

$$= \frac{1}{a} \|x\| \qquad \because \|x\| = a$$

$$= \frac{a}{a} = 1$$

So $\|y\| = 1$

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{\|T(x)\|}{\|x\|} \right\}$$

$$= \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \frac{\|T(x)\|}{a} \right\}$$ since $\|x\| = a$

$$= \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \left\| \frac{1}{a} T(x) \right\| \right\}$$ property of norm

$$= \sup_{\substack{x \in X \\ x \neq 0}} \left\{ \left\| T\left(\frac{1}{a} x\right) \right\| \right\}$$ T is linear operator

$$= \sup_{\substack{y \in X \\ \|y\| = 1}} \left\{ \|T(y)\| \right\}$$

$$\therefore \|T\| = \sup_{\substack{y \in X \\ \|y\| = 1}} \left\{ \|T(y)\| \right\}.$$

#

⑥ differentiation Operator

Let X be the normed space of all polynomials on $J = [0, 1]$ with norm given

$$\|x\| = \max\{|x(t)|,\ t \in J\}$$

where $x(t) = 1 + t + t^2 + \dots\dots t^n$, $n \in \mathbb{N}$

A differentiation operator T is ~~denoted~~ defined on X by $T : X \longrightarrow X$

$$T(x)(t) = x'(t) \qquad t \in J$$

where the prime denotes differentiation with respect to t. This operator is linear but not bounded. ~~Inde.~~

H.L.

Indeed. Let $(x_n)_{n=1}^{\infty}$ be a set of Polynomials $x_n(t) = t^n$, where $n \in \mathbb{N}$, ~~it is clearly that~~

So

$$\|x_n\| = \max\{|x(t)|,\ t \in [0, 1]\}$$

$$= \max\{|t^n|,\ t \in [0, 1]\}$$

$$= 1 \qquad \forall n \in \mathbb{N}$$

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$T(x_n)(t) = x'(t) = (t^n)'$
$= n t^{n-1}$

$\| T(x_n)(t) \| = \cancel{\max \{ \ldots T(x_n)(t) \}}$
$= \| n t^{n-1} \|$
$= \max \{ | n t^{n-1} |, t \in [0,1] \}$
$= n \max \{ | t^{n-1} | \, t \in [0,1] \}$
$= n$

$$\frac{\| T(x_n)(t) \|}{\| x_n \|} = \frac{n}{1} = n$$

for all $c \in R^*$ not exist element such that

$$\boxed{n \leq c \quad \forall n \in N}$$

not exists

~~so~~ we see for some $x_n(t) \in X$, has upper bounded n which is not bounded when n is ~~[illegible]~~ large ~~[illegible]~~ So

T is not bounded.

8 Integral operator.

Let $C[0,1]$ be a normed space of all continuous function on interval $[0,1]$ with the norm.

$$\|f\| = \max_{0 \le t \le 1} \{|f(t)|\}$$

for $f \in C[0,1]$.

we can define an integral operator.

$T : C[0,1] \longrightarrow C[0,1]$ by

$$\big(T(f)\big)(t) = \int_0^1 k(t,x)\, f(x)\, dx = g(t)$$

That is mean $T(f) = g$

where $f, g \in C[0,1]$ and

k is a given function which is called the Kerenal of T and is assumed to be continuous on the closed square

$G = [0,1] \times [0,1]$

This operator is linear and bounded.

At first we to show that T is linear operator so, let $f, g \in C[0,1]$, then

$$(T(f+g))(t) = \int_0^1 k(t,x)\,(f+g)(x)\,dx$$

$$= \int_0^1 k(t,x)\,(f(x)+g(x))\,dx$$

$$= \int_0^1 k(t,x) f(x)\,dx + \int_0^1 k(t,x) f(x)\,dx$$

$$= (T(f))(t) + (T(g))(t)$$

$$= (T(f)+T(g))(t) \qquad \forall t \in [0,1]$$

$$\therefore T(f+g) = T(f) + T(g)$$

Let $\alpha \in R$, $f \in C[0,1]$

$$T(\alpha f)(t) = \int_0^1 k(t,x)\,(\alpha f)(x)\,dx$$

$$= \int_0^1 k(t,x)\,\alpha f(x)\,dx$$

$$= \alpha \int_0^1 k(t,x) f(x)\,dx$$

$$= \alpha\, T(f)(t) \qquad \forall t \in [0,1]$$

$$\therefore T(\alpha f) = \alpha\, T(f)$$

$\therefore T$ is linear operator.

(10) . To prove that T is bounded, we start at first with the function $k(t,x)$ which is a continuous at $G = [0,1] \times [0,1]$ by using extrem value theorem

" Each function of real number wich is continuous on closed interval has maximum and minimum "

That is mean $k(t,x)$ is bounded let $c>0$ is the bounded for the function $k(t,x)$.

$$|k(t,x)| \leq c \qquad \forall (t,x) \in G \quad \text{—— (1)}$$

$$\because |f(t)| \leq \max_{0 \leq t \leq 1} \{ |f(t)| \} = \|f\| \quad \text{—— (2)}$$

now

$$\|T(f)\| = \max_{0 \leq t \leq 1} \left\{ \left| \int_0^1 k(t,x) f(x) dx \right| \right\}$$

$$\leq \max_{0 \leq t \leq 1} \left\{ \int_0^1 |k(t,x)| \, |f(x)| \, dt \right\}$$

$$\leq \max_{0 \leq t \leq 1} \left\{ \int_0^1 c \, \|f\| dx \right\} \quad (2) \& (1) \text{ in}$$

$$= c \|f\| \max_{0 \leq t \leq 1} \left\{ \int_0^1 dx \right\}$$

$$= c \|f\| \max_{0 \leq t \leq 1} \{ 1 \}$$

$$= c \|f\| \qquad \forall f \in c[0,1]$$

$\therefore$ T is bounded with constant c

matrix operator

Let $A = [a_{ij}]_{r \times n}$ be a matrix with r rows and n columns defines an operator

$T : R^n \longrightarrow R^r$ by means of

$T(x) = Ax = y$

where $x \in R^n$, $x = (x_1, x_2, \cdots, x_n)$

$y \in R^r$ $y = (y_1, y_2, \cdots, y_r)$

with $\|x\| = \left(\sum_{i=1}^{n} (x_i)^2 \right)^{1/2}$

$\|y\| = \left(\sum_{i=1}^{r} y_i^2 \right)^{1/2}$

and the element in $y_i = \sum_{j=1}^{n} a_{ij} x_j$

$i = 1, 2, \dots, r$

At first we to show that T is linear operator.

Let $x, z \in R^n$, so

$T(x+y) = A(x+z)$

$= Ax + Az$

$= T(x) + T(z)$