Example Let $X = \mathbb{C}^2$ and $Y = \cancel{M_2(\mathbb{C})}$ $\mathbb{C}$ ~~matrix~~

when $x \in \mathbb{C}^2$

$x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{C}$

we define $h : X \times Y \longrightarrow \mathbb{C}$

$$h(x,y) = h((x_1,x_2), y) = x_1\bar{y} + x_2\bar{y}$$

show that h is sesquilinear form

Solution Let $v, w \in X$, $\alpha, \beta \in \mathbb{C}$

$y \in Y$

$v = (v_1, v_2)$

$w = (w_1, w_2)$

$$\begin{aligned}
h(v+w, y) &= h((v_1,v_2)+(w_1,w_2), y) \\
&= h((v_1+w_1, v_2+w_2), y) \\
&= (v_1+w_1)\bar{y} + (v_2+w_2)\bar{y} \\
&= v_1\bar{y} + w_1\bar{y} + v_2\bar{y} + w_2\bar{y} \\
&= v_1\bar{y} + v_2\bar{y} + w_1\bar{y} + w_2\bar{y} \\
&= h((v_1,v_2),y) + h((w_1,w_2),y) \\
&= h(v,y) + h(w,y)
\end{aligned}$$

$$
\begin{aligned}
h(\alpha v, y) &= h(\alpha(v_1, v_2), y) \\
&= h((\alpha v_1, \alpha v_2), y) \\
&= \alpha v_1 \overline{y} + \alpha v_2 \overline{y} \\
&= \alpha (v_1 \overline{y} + v_2 \overline{y}) \\
&= \alpha\, h((v_1, v_2), y) \\
&= \alpha\, h(v, y)
\end{aligned}
$$

for $x, y \in \mathbb{C}$

$$
\begin{aligned}
h(v, x+y) &= h((v_1, v_2), x+y) \\
&= v_1 \overline{x+y} + v_2 \overline{x+y} \\
&= v_1 (\overline{x} + \overline{y}) + v_2 (\overline{x} + \overline{y}) \\
&= v_1 \overline{x} + v_1 \overline{y} + v_2 \overline{x} + v_2 \overline{y} \\
&= v_1 \overline{x} + v_2 \overline{x} + v_1 \overline{y} + v_2 \overline{y} \\
&= h((v_1, v_2), x) + h((v_1, v_2), y) \\
&= h(v, x) + h(v, y)
\end{aligned}
$$

$$
\begin{aligned}
h(v, \alpha y) &= h((v_1, v_2), \alpha y) \\
&= v_1 \overline{\alpha y} + v_2 \overline{\alpha y} \\
&= v_1 \overline{\alpha}\, \overline{y} + v_2 \overline{\alpha}\, \overline{y} \\
&= \overline{\alpha} (v_1 \overline{y} + v_2 \overline{y}) \\
&= \overline{\alpha}\, h((v_1, v_2), y) \\
&= \overline{\alpha}\, h(v, y)
\end{aligned}
$$

So h is sesquilinear form

IS h bounded operator

Let $v \in \mathbb{C}^2$ such that $v = (v_1, v_2)$

$y \in \mathbb{C}$

$$|h(v,y)| = |h((v_1,v_2),y)|$$
$$= |v_1\bar{y} + v_2\bar{y}|$$
$$= |\langle (v_1,v_2)/(y,y) \rangle| \quad \text{by } \cancel{\text{Cauchy}} \text{ Shawarz inquality}$$
$$\leq \|(v_1,v_2)\| \, \|(y,y)\|$$
$$= \|v\| \sqrt{\langle (y,y)/(y,y) \rangle}$$
$$= \|v\| \sqrt{y\bar{y} + y\bar{y}}$$
$$= \|v\| \sqrt{2y\bar{y}}$$
$$= \|v\| \sqrt{2|y|^2}$$
$$= \|v\| \sqrt{2}\, |y|$$

$\Rightarrow |h(v,y)| \leq \sqrt{2}\, \|v\| \, |y|$

$\forall v \in \mathbb{C}^2, y \in \mathbb{C}$

$\Rightarrow h$ is bounded with Constant $= \sqrt{2}$

Theorem (Riesz representation)

Let X, Y be two Hilbert spaces
$h : X \times Y \longrightarrow K$ functional
be a bounded sesquilinear form. Then
h has a representation

$$h(x,y) = \langle S(x), y \rangle$$

where $S : X \longrightarrow Y$ is a bounded linear operator
S is uniquely determined by h and has norm

$$\|S\| = \|h\|.$$

Example Let $X = \mathbb{C}^2$, $Y = \mathbb{C}$

$$h(v, y) = h((v_1, v_2), y) = v_1 \bar{y} + v_2 \bar{y}.$$

we explain that h is bounded sesquilinear form.

$$\begin{aligned} h(v, y) &= h((v_1, v_2), y) \\ &= v_1 \bar{y} + v_2 \bar{y} \\ &= (v_1 + v_2)\bar{y} \\ &= \cancel{\langle v_1 + v_2 / y \rangle} \\ &= \langle v_1 + v_2 / y \rangle \end{aligned}$$

So ~~we~~ define $S : X = \mathbb{C}^2 \longrightarrow Y = \mathbb{C}$

$S(v) = S(v_1, v_2) = v_1 + v_2$

so must S is bounded linear operator

for $v, w \in \mathbb{C}^2$, $\alpha, \beta \in \mathbb{C}$

$$S(\alpha v + \beta w) = S(\alpha(v_1, v_2) + \beta(w_1, w_2))$$
$$= S((\alpha v_1, \alpha v_2) + (\beta w_1, \beta w_2))$$
$$= S(\alpha v_1 + \beta w_1, \alpha v_2 + \beta w_2)$$
$$= \alpha v_1 + \beta w_1 + \alpha v_2 + \beta w_2$$
$$= \alpha v_1 + \alpha v_2 + \beta w_1 + \beta w_2$$
$$= \alpha(v_1 + v_2) + \beta(w_1 + w_2)$$
$$= \alpha S(v_1, v_2) + \beta S(w_1, w_2)$$
$$= \alpha S(v) + \beta S(w)$$

$\therefore S$ is linear operator and it is bounded

$$\cancel{\|S(v)\| = \|S(v_1, v_2)\|}$$
$$\cancel{= \|v_1 + v_2\|}$$