Divergence and Circulation
التباعد والدوران

Definition: If $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ then we define the divergence of F written **div F** by:

$$\text{div F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Definition: Let $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ then we defined the circulation of F written **Curl F** by:

$$Curl\ F = \left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right] i + \left[\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right] j + \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] k$$

Where

$Curl\ F = \nabla \times F$, $\nabla = \frac{\partial}{\partial x}\ i + \frac{\partial}{\partial y}\ j + \frac{\partial}{\partial z}\ k$

and

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right] i - \left[\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right] j + \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] k$$
$$= \left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right] i + \left[\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right] j + \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] k$$

Example 1: Find the **div F** and **curl F** of the vector field

$$F(x,y,z) = x^2 y\ i + 2\ y^3 z\ j + 3z\ k$$

Solution:

$div\ F = 2xy + 6y^2 z + 3$

$curl\ F = (0 - 2\ y^3)i + (0 - 0)j + (0 - x^2)k$

$\therefore\ \ curl\ F = -2\ y^3\ i\ -\ x^2\ k$

Example 2: Find the **div F** of $F(x,y) = (x^2 - y)\ i + (xy - y^2)\ j$

$div\ F = \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y}$

$div\ F = 2x + x - 2y = 3x - 2y$

Definition: The Laplacian ∇^2

The operator that results by taking the dot product with itself denoted by ∇^2 and is called Laplacian operator, then operator has the form

$$\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k\right) \cdot \left(\frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Note: 1. If $\emptyset(x, y, z)$**,** the Laplacian operator product the function

$$\nabla^2 \emptyset = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2}$$

If equation $\nabla^2 \emptyset = 0$ i.e., $\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} + \frac{\partial^2 \emptyset}{\partial z^2} = 0$ that is known by Laplace equation**.**

2. $\nabla^2 \emptyset = div\,(\nabla \emptyset)$.

Example 3: If $F(x, y, z) = sinx\; i + \cos(x - y)\; j + z\; k$

Find (1) the operator $\nabla^2 F$ (2) $\nabla \cdot (\nabla \times \mathrm{F})$

Solution:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\frac{\partial F}{\partial x} = cosx - \sin(x - y)$, $\frac{\partial^2 F}{\partial x^2} = -sin\, x - \cos(\mathrm{x} - \mathrm{y})$

$\frac{\partial F}{\partial y} = \sin(x - y)$, $\frac{\partial^2 F}{\partial y^2} = - \cos(\mathrm{x} - \mathrm{y})$

$\frac{\partial F}{\partial z} = 1$, $\frac{\partial^2 F}{\partial y^2} = 0$

$$\nabla^2 F = -sin\, x - \cos(\mathrm{x} - \mathrm{y}) - \cos(\mathrm{x} - \mathrm{y}) = -sin\, x - 2\cos(\mathrm{x} - \mathrm{y})$$

(2) $$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ sinx & \cos(x - y) & z \end{vmatrix} =$$

$$= (0 - 0)i - (0 - 0)j + (-\sin(x - y) - 0)k$$

$$\therefore\ \nabla \times F = -\sin(x - y)\, k$$

$$\nabla \cdot (\nabla \times F) = \left(\frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k\right) \cdot -\sin(x - y)\, k = 0$$

$$\therefore\ \nabla \cdot (\nabla \times F) = 0$$

Example 4: If $F(x,y,z) = 2x\, i + \, j + 4y\, k$ and
$G(x,y,z) = x\, i + \, yj - z\, k$, find $\nabla .\, (\mathrm{F} \times \mathrm{G})$

Solution:

$$\mathrm{F} \times \mathrm{G} = \begin{vmatrix} i & j & k \\ 2x & 1 & 4y \\ x & \mathrm{y} & -z \end{vmatrix} =$$

$$= (-z - 4y^2)i - (-2xz - 4xy)j + (2xy - x)k$$

$$\nabla .\, (\mathrm{F} \times \mathrm{G}) = \left(\frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k \right) (-z - 4y^2)i - (-2xz - 4xy)j + (2xy - x)k$$

$$\therefore \quad \nabla .\, (\mathrm{F} \times \mathrm{G}) = 0 + 4x + 0 = 4x$$